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Reasoning with Fuzzy Extensions of OWL and OWL 2

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Abstract Fuzzy Description Logics (f-DLs) have been proposed as logical formalisms capable of representing and reasoning with vague/fuzzy information. They are envisioned to be helpful for many applications that need to cope with such type of information such as multimedia processing, decision making, automatic negotiation and more. Recent results have provided with many tableaux algorithms for supporting reasoning over quite expressive f-DLs. However, no (direct) tableaux algorithm for reasoning with fuzzy extensions of DLs such as $\mathcal{SHOIQ}$ and $\mathcal{SROIQ}$ exist today. $\mathcal{SHOIQ}$ and $\mathcal{SROIQ}$ are particularly interesting formalisms as they constitute the logical underpinnings of the Web ontology languages OWL DL and OWL 2 DL. In the current paper we present an algorithm for reasoning with the fuzzy DLs f-$\mathcal{SHOIQ}$ and f-$\mathcal{SROIQ}$. In addition, we also provide provide a tableaux algorithm for fuzzy nominals, thus providing reasoning support for the fuzzy DL language (we call) f-$\mathcal{SHO}_{\mathfrak{f}}\mathcal{IQ}$.

Keywords Fuzzy Description Logics · Reasoning · Fuzzy Nominals · fuzzy-$\mathcal{SHOIQ}$ · fuzzy-$\mathcal{SROIQ}$

1 Introduction

Description Logics (DLs) [1] are a family of class-based (concept-based) logical formalisms equipped with well-defined model-theoretic semantics. Nowadays, DLs have gained considerable attention due to their application in the context of the Semantic Web. More precisely, both W3C's standards for representing ontologies in the (Semantic) Web—that is, OWL DL [28] and OWL 2 DL [38], are based on the very expressive Description Logics $\mathcal{SHOIN}(\mathbf{D}^+)$ [30] and $\mathcal{SROIQ}(\mathbf{D}^+)$ [31], respectively. Furthermore, the (robust) decidability of DLs and the existence of practically efficient and scalable reasoning systems has

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made them attractive to researchers from many research areas who have applied them in domains like, bio-medical information systems [42,23,40], astronomy [19], defence [34] and more.

Since classical DLs do not provide means for representing fuzzy/vague information, fuzzy DLs [36,50,51] and fuzzy OWL [12,47] have been proposed, which extend classical DLs and OWL using the theory of fuzzy sets [33]. Using fuzzy DLs one can directly represent *fuzzy* concepts such as 'Tall', 'Blue', 'Hot', 'Large', for which a clear and precise definition is difficult (if not impossible). Fuzzy DLs are also gaining significant attention and have been used in many applications due to the fact that they combine the flexibility of fuzzy sets with the formality and scalability of Description Logics. Examples of such applications are multimedia processing and retrieval [17,37,43], semantic portals [26], ontology mapping [21], decision making [54] and negotiation [9].

Several theoretical works have been carried out towards providing reasoning support for expressive fuzzy DL languages. More precisely, Straccia [51] presented the first tableaux-based reasoning algorithm for f_{KD}-$\mathcal{ALC}$. f_{KD}-$\mathcal{ALC}$ is the fuzzy-$\mathcal{ALC}$ DL that uses specific fuzzy operators, namely min for conjunction, max for union, and $1-x$ for negation, also called standard fuzzy operators. Later Stoilos et al. [50] extended this algorithm for the fuzzy DLs f_{KD}-$\mathcal{SI}$ and f_{KD}-$\mathcal{SHIN}$. Also recently a tableaux algorithm for $\mathcal{SHI}$ with semantics based on a complete residuated finite De Morgan lattice has been presented [14]. In a different approach, Bobillo et al. [6,7,11] have presented techniques that translate fuzzy DL knowledge bases to classical DLs. The motivation of this approach is to use existing classical DLs to provide reasoning support for fuzzy DLs. However, the reduced ontology is quadratically larger than the input fuzzy one, mainly due to additional axioms that are required in order to capture the relations between the fuzzy degrees. Hence, on the one hand, preliminary evaluation has shown that these additional axioms do affect performance in practice [16]. On the other hand, direct tableaux-based techniques are amenable to direct (internal) optimisations techniques [44], hence tableaux-based algorithms are still relevant for fuzzy DL research. Moreover, there have been many recent undecidability results for fuzzy DLs that use operators other than the standard fuzzy ones [2,3,13,15]. Hence, one can still advocate in favour of studying fuzzy DLs under the standard fuzzy operators.

Although tableaux algorithms for quite expressive fuzzy DLs have been presented there is still no tableaux reasoning algorithm that supports nominals (denoted in DLs by $\mathcal{O}$). Nominals is an important feature of OWL DL as one could use them to specify enumerations of elements. For example, we can define the concept of European Countries as follows:

$$\mathsf{EUCountries} \equiv \{greece\} \sqcup \{germany\} \sqcup \ldots \sqcup \{italy\}$$

where $\mathsf{EUCountries}$ is a *concept* (unary predicate), $greece, germany, \ldots, italy$ are *individuals* (objects) and $\equiv$ is an equivalence axiom. In the current paper we first contribute to the state-of-the-art of fuzzy DLs by extending the reasoning algorithm of f_{KD}-$\mathcal{SHIN}$ [50] in order to support nominals. Additionally, we also integrate the results for tableaux reasoning with *qualified cardinality restrictions* (denoted in DLs by $\mathcal{Q}$) presented in [45], thus we present a reasoning algorithm for the fuzzy DL f_{KD}-$\mathcal{SHOIQ}$. To the best of our knowledge this is the first tableaux algorithm for f_{KD}-$\mathcal{SHOIQ}$, i.e. for f_{KD}-OWL extended with qualified cardinality restrictions.[1]

The reasoning algorithm presented for f_{KD}-$\mathcal{SHOIQ}$ treats nominals in a crisp (non-fuzzy) way, based on the semantics presented in [47]. However, Bobillo et al. [5] presented a fuzzy extension of nominals, creating *fuzzy nominals*. Fuzzy nominals are defined by $\{o, n\}$,

[1] Note that a preliminary and incomplete account for nominals appeared in [47].

where o is an individual and $n \in (0,1]$. The intuition of fuzzy nominals is quite similar to that of classical nominals. More precisely, one can explicitly enumerate the members of a *fuzzy* set, thus also the membership degrees of objects need to be specified. For example, using fuzzy nominals one is able to define the concept of Vulcan Mediterranean Countries as:

$$\mathsf{VulcanMed} \equiv \{(greece,1)\} \sqcup \{(albania,0.8)\} \sqcup \{(montenegro,0.7)\} \sqcap \{(croatia,0.6)\}$$

saying that Albania, Montenegro and Croatia are to some extend Mediterranean countries. Although the semantics of fuzzy nominals were presented in [5], no direct tableaux reasoning algorithm was given. In the current paper we will extent the algorithm of f_{KD}-$\mathcal{SHOIQ}$ in order to also provide reasoning support for fuzzy nominals.

In addition, there is currently no direct tableaux reasoning algorithm for reasoning with fuzzy extensions of OWL 2 DL. OWL 2 DL extends OWL DL with new concept constructors and many new role axioms that can be particularly usefull for many applications that use fuzzy DLs, like multimedia analysis. More precisely, with OWL 2 DL we can describe the fact that the (fuzzy) role containsRegion is *irreflexive* and *antisymmetric* or capture complex partonomic and spatial relations between multimedia objects with the aid of *complex role inclusions*, like the following:

$$\mathsf{isAboveOf} \circ \mathsf{isRightOf} \sqsubseteq \mathsf{isAboveRightOf}$$

which states that a region which is above and right of some other region in an image, then it is also above-right,

In summary the current paper makes the following major contributions:

- It presents a tableaux reasoning algorithm for supporting nominals in fuzzy Description Logics. It then combines this algorithm with the algorithm about qualified cardinality restrictions [45] and General Concept Inclusion axioms [48], thus presenting a reasoning algorithm for the fuzzy DL language f_{KD}-$\mathcal{SHOIQ}$. This language is particularly important since, discarding datatypes, it consists to the f_{KD}-OWL fuzzy ontology language [49].
- It extends the reasoning algorithm of f_{KD}-$\mathcal{SHOIQ}$ in order to provide reasoning support for fuzzy nominals [5] for which, to the best of our knowledge, no tableaux reasoning algorithm exists in the literature.
- It further extends these tableaux algorithms presenting a tableaux reasoning algorithm for f_{KD}-$\mathcal{SROIQ}$, a fuzzy extension of the $\mathcal{SROIQ}$ DL. To do so, we first extend a central result from [32] which shows that the same automata technique used in [32] to capture the semantics of complex role inclusions can also be used in the case of f_{KD}-$\mathcal{SROIQ}$. Second, we provide a technique by which complex role inclusion axioms can be encoded as normal f_{KD}-$\mathcal{SHOIQ}$ axioms following the results in [18]. Consequently, our tableaux algorithm for f_{KD}-$\mathcal{SROIQ}$ only has minor extensions compared to the one presented for f_{KD}-$\mathcal{SHOIQ}$. This algorithm is particularly important since given the previous transformations we can provide reasoning support for f_{KD}-OWL 2 DL.

The rest of the paper is organized as follows. In Section 2.1 we give a quick look to the classical $\mathcal{SHOIQ}$ language which will later be extended using notions from fuzzy set theory, thus creating the fuzzy DLs f_{KD}-$\mathcal{SHOIQ}$ and f_{KD}-$\mathcal{SHO}_{\mathfrak{f}}\mathcal{IQ}$, while in Section 2.2 we give a brief introduction to fuzzy set theory. In Section 3 we first present the syntax and semantics of f$_{KD}$-$\mathcal{SHOIQ}$ and its reasoning problems, while later we present the fuzzy DL f$_{KD}$-$\mathcal{SHO}_{\mathfrak{f}}\mathcal{IQ}$. Then, in Section 4 we present a reasoning algorithm that decides the key inference problems of f$_{KD}$-$\mathcal{SHOIQ}$ and prove its correctness. Subsequently, in Section 5 we present all the necessary extensions required on the f$_{KD}$-$\mathcal{SHOIQ}$ algorithm to support fuzzy nominals. Finally, in Section 6 we extend this algorithm even further in order to provide a reasoning algorithm for the fuzzy DL f$_{KD}$-$\mathcal{SROIQ}$, while Section 7 concludes the paper. Proofs of key results are given in an Appendix.

2 Preliminaries

2.1 The Description Logic $\mathcal{SHOIQ}$

In this section, we will briefly introduce the $\mathcal{SHOIQ}$ DL, which will be extended later.

A description language consists of an alphabet of distinct *concept names* (or atomic concepts) (**C**), *role names* (**R**) and *individual names* (or individuals) (**I**) together with a set of constructors to construct concept and role descriptions. Let $RN \in \mathbf{R}$ be a role name and R a $\mathcal{SHOIQ}$-role. $\mathcal{SHOIQ}$-roles are defined (inductively) by the syntax $S \to RN \mid R^-$, where R^- represents the *inverse* of R. The inverse relation of roles is symmetric and to avoid considering roles such as R^{--} we define the function Inv which returns the inverse of a role. More precisely, $\mathsf{Inv}(R) := RN^-$ if $R = RN$, while $\mathsf{Inv}(R) := RN$ if $R = RN^-$. Let A be a concept name, R a $\mathcal{SHOIQ}$-role, S a *simple* $\mathcal{SHOIQ}$-role,[2] $p \in \mathbb{N}$ and $o \in \mathbf{I}$. $\mathcal{SHOIQ}$-concepts are defined inductively as follows:

$$C,D \;\longrightarrow\; \top \mid \bot \mid A \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid \exists R.C \mid \forall R.C \mid \geq pS.C \mid \geq pS.C \mid \{o\}$$

Note that the concept $= pS.C$ can be used as an abbreviation of $\geq pS.C \sqcap \leq pS.C$.

Description Logics have a model theoretic semantics, which are defined in terms of interpretations. An *interpretation* $\mathcal{I}$ consists of nonempty set of objects $\Delta^{\mathcal{I}}$ called *domain*, and an *interpretation function* $\cdot^{\mathcal{I}}$ which maps each individual name $a \in \mathbf{I}$ to an object $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, each concept name $A \in \mathbf{C}$ to a subset $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, and each role name $RN \in \mathbf{R}$ to a binary relation $RN^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. Furthermore, for $x,y \in \Delta^{\mathcal{I}}$, $\langle x,y\rangle \in R^{\mathcal{I}}$ if and only if (iff) $\langle y,x\rangle \in (R^-)^{\mathcal{I}}$. The interpretation of $\mathcal{SHOIQ}$-concepts is presented in Table 1, where $x,y \in \Delta^{\mathcal{I}}$.

A $\mathcal{SHOIQ}$ *TBox* $\mathcal{T}$ is a finite set of *concept inclusion* (also called *subsumption*) axioms of the form $C \sqsubseteq D$, where C,D are $\mathcal{SHOIQ}$-concepts. We also often write $C \equiv D$ instead of $C \sqsubseteq D$ and $D \sqsubseteq C$. An interpretation $\mathcal{I}$ satisfies $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. Note that concept inclusion axioms of this form are called *General Concept Inclusions* (GCIs) [1].

A $\mathcal{SHOIQ}$ *RBox* $\mathcal{R}$ is a finite set of *transitive role* axioms of the form $\mathsf{Trans}(R)$, and *Role Inclusion Axioms* (RIAs) of the form $R \sqsubseteq S$. An interpretation $\mathcal{I}$ satisfies $\mathsf{Trans}(R)$ if, for all $x,y,z \in \Delta^{\mathcal{I}}$, $\{\langle x,y\rangle, \langle y,z\rangle\} \subseteq R^{\mathcal{I}} \to \langle x,z\rangle \in R^{\mathcal{I}}$, and it satisfies $R \sqsubseteq S$ if $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$. A set of role inclusion axioms defines a *role hierarchy*. For a role hierarchy we denote by $\sqsubseteq^*$ the transitive-reflexive closure of $\sqsubseteq$ over the set $\{R \sqsubseteq S, \mathsf{Inv}(R) \sqsubseteq \mathsf{Inv}(S) \mid R \sqsubseteq S \in \mathcal{R}\}$. At last, observe that if $R \sqsubseteq S$, then the semantics of role inclusion axioms imply that $\mathsf{Inv}(R)^{\mathcal{I}} \subseteq \mathsf{Inv}(S)^{\mathcal{I}}$.

A $\mathcal{SHOIQ}$ *ABox* $\mathcal{A}$ is a finite set of individual axioms (or assertions) of the form $a : C$, called *concept assertions*, or $(a,b) : R$, called *role assertions*, or $a \not\doteq b$, stating that two

[2] A role is called *simple* if it is neither transitive nor it has any transitive subrole. This is important to ensure decidability [30].

Table 1: Semantics of $\mathcal{SHOIQ}$-concepts

Constructor	Syntax	Semantics
top	$\top$	$\Delta^{\mathcal{I}}$
bottom	$\bot$	$\emptyset$
general negation	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
disjunction	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
exists restriction	$\exists R.C$	$\{x \mid \exists y.\langle x,y\rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$
value restriction	$\forall R.C$	$\{x \mid \forall y.\langle x,y\rangle \in R^{\mathcal{I}} \to y \in C^{\mathcal{I}}\}$
at-most	$\leq pS.C$	$\{x \mid \sharp\{y \mid S^{\mathcal{I}}(x,y) \wedge C^{\mathcal{I}}(y)\} \leq p\}$
at-least	$\geq pS.C$	$\{x \mid \sharp\{y \mid S^{\mathcal{I}}(x,y) \wedge C^{\mathcal{I}}(y)\} \geq p\}$
nominal	$\{o\}$	$\{o\}^{\mathcal{I}} = \{o^{\mathcal{I}}\}$

individuals are different. An interpretation $\mathcal{I}$ satisfies $a : C$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$, it satisfies $(a,b) : R$ if $\langle a^{\mathcal{I}}, b^{\mathcal{I}}\rangle \in R^{\mathcal{I}}$ and it satisfies $a \not\doteq b$, if $a^{\mathcal{I}} \neq b^{\mathcal{I}}$.

Finally, a $\mathcal{SHOIQ}$ knowledge base (KB) consists of a tuple $\langle \mathcal{T}, \mathcal{R}, \mathcal{A}\rangle$ where $\mathcal{T}$ is a TBox $\mathcal{R}$ is an RBox, and $\mathcal{A}$ is an ABox.

2.2 Fuzzy Sets

Fuzzy set theory is an *extension* of classical set theory [55]. While in crisp sets an object either belongs (to a degree 1) or not (to a degree 0) to a crisp set S, in fuzzy set theory this notion is extended in a way such that an object belongs to a fuzzy set to any degree between 0 and 1. This membership degree is defined by a *membership function* of the form $\mu_A : X \to [0,1]$, or simply $A : X \to [0,1]$, where X is the universal set and A is a fuzzy subset of X. Then, given an object $x \in X$, μ_A returns the membership degree of x to the fuzzy set A. For example we can say that *John* belongs to the set of *Tall* people to a degree of 0.7, writing $Tall(John) = 0.7$. Similarly a fuzzy relation R is defined by a function of the form $R : X \times X \to [0,1]$.

The classical set theoretic operations of complement, union, intersection and the logical implication are also extended to the framework of fuzzy set theory and logic [33]. In the new context they are performed by special mathematical functions over the unit interval called *triangular norm* operations [33]. Though there is a wealth of norm operations defined in the literature, reasoning with arbitrary such operators in fuzzy DLs is an extremely difficult task [25] and has led to many undecidability results [2,3,13,15]. Thus, in the current paper we focus only on specific norms, namely on the Lukasiewicz negation, given by function $c(a) = 1 - a$, the Gödel t-norm for conjunction, given by function $t(a,b) = \min(a,b)$, the Gödel t-conorm for disjunction, given by function $u(a,b) = \max(a,b)$, and the Kleene-Dienes fuzzy implication, given by function $\mathcal{J}(a,b) = \max(1-a,b)$.

From a mathematical point of view, these functions satisfy the following properties. The Lukasiewicz negation satisfies the *boundary conditions*, $c(0) = 1$ and $c(1) = 0$, is *monotonic decreasing*, for $a \leq b$, $c(a) \geq c(b)$, *continuous* and finally *involutive*—that is, for each $a \in [0,1]$ we have $c(c(a)) = a$. The Gödel t-norm (t-conorm) satisfies the standard properties of norm operation, namely the *boundary condition*, $t(a,1) = a$ $(u(a,0) = a)$, is *monotonic increasing*, for $b \leq d$ then $t(a,b) \leq t(a,d)$ $(u(a,b) \leq u(a,d))$, *commutative*, $t(a,b) = t(b,a)$ $(u(a,b) = u(b,a))$, and *associative*, $t(a,t(b,c)) = t(t(a,b),c)$ $(u(a,u(b,c)) = u(u(a,b),c))$.

Moreover, due to the boundary conditions it holds that $t(a,0) = 0$ ($u(a,0) = a$). Additionally, it is the only t-norm (t-conorm) that is *idempotent*, i.e., $\min(a,a) = a$ ($\max(a,a) = a$). Now, we recall a property of the max operator that we are going to use in Section 6. For any $a,b \in [0,1]$, where j takes values from the index set J, the max operation satisfies the following Property:

$$(\blacklozenge) : \inf_{j \in J} \max(a, b_j) = \max(a, \inf_{j \in J} b_j)$$

Finally, we also recall some properties and notions for fuzzy relations. A fuzzy relation R over $X \times X$ is called sup-t *transitive*, or simply *transitive* if $\forall a,b \in X, R(a,c) \geq \sup_{b \in X}\{t(R(a,b),R(b,c))\}$. R is *reflexive* if $\forall a \in X, R(a,a) = 1$, while it is called *irreflexive* if $\forall a \in X, R(a,a) = 0$.[3] The *inverse* of a fuzzy relation $R : X \times Y \to [0,1]$ is a fuzzy relation $R^- : Y \times X \to [0,1]$ defined as $R^-(b,a) = R(a,b)$. Finally, given two fuzzy relations $R_1 : X \times Y \to [0,1]$ and $R_2 : Y \times Z \to [0,1]$ we define the sup-t composition as, $[R_1 \circ^t R_2](a,c) = \sup_{b \in Y}\{t(R(a,b),R(b,c))\}$. The operation of sup-$t$ composition satisfies the following properties:

$$(R_1 \circ^t R_2) \circ^t R_3 = R_1 \circ^t (R_2 \circ^t R_3), \qquad (R_1 \circ^t R_2)^- = (R_2^- \circ^t R_1^-)$$

Due to the associativity property we can extend the operation of sup-t composition to any number of fuzzy relations; hence, we will simply write $[R_1 \circ^t R_2 \circ^t \ldots \circ^t R_n](a,b)$.

3 The fuzzy DLs f$_{KD}$-$\mathcal{SHOIQ}$ and f$_{KD}$-$\mathcal{SHO}_\mathtt{f}\mathcal{IQ}$

In this section we present the fuzzy DLs f$_{KD}$-$\mathcal{SHOIQ}$ and f$_{KD}$-$\mathcal{SHO}_\mathtt{f}\mathcal{IQ}$. The former extends the syntax of $\mathcal{SHOIQ}$ with membership degrees in concept and role assertions, thus adding fuzziness at the instance level. The latter extends f$_{KD}$-$\mathcal{SHOIQ}$ with the *fuzzy nominal constructor* [5], which is defined by adding fuzziness on a specific concept constructor. The syntax and semantics of these fuzzy DLs have been presented before in various works [52,46,5]; we recall them here for the sake of completeness. Additionally we introduce some useful simplifications that would make the presentation of the algorithms easier and finally show how the inference problems can be reduced to KB satisfiability.

3.1 The Fuzzy DL f$_{KD}$-$\mathcal{SHOIQ}$

As with classical DLs we have an alphabet of distinct concept names, role names, and individual names. f$_{KD}$-$\mathcal{SHOIQ}$-roles and f$_{KD}$-$\mathcal{SHOIQ}$-concepts are defined in the same way as their crisp $\mathcal{SHOIQ}$ counterparts. The semantics, however, are provided by the means of *fuzzy interpretations* [51]. A fuzzy interpretation is a pair $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ where the domain $\Delta^\mathcal{I}$ is a non-empty set of objects and $\cdot^\mathcal{I}$ is a fuzzy interpretation function which maps:

- an individual a to an element of $a^\mathcal{I} \in \Delta^\mathcal{I}$,
- a concept name A to a membership function $A^\mathcal{I} : \Delta^\mathcal{I} \to [0,1]$, and
- a role name R to a membership function $R^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to [0,1]$.

[3] Note that in most fuzzy textbooks this property is referred to as antireflexivity [33], but in order to be aligned with the semantics of OWL 2 DL [31] we call it irreflexivity.

Table 2: Semantics of f_{KD}-$\mathcal{SHOIQ}$-concepts and f_{KD}-$\mathcal{SHOIQ}$-roles

Constructor	Semantics
top	$\top^{\mathcal{I}}(a) = 1$
bottom	$\bot^{\mathcal{I}}(a) = 0$
general negation	$(\neg C)^{\mathcal{I}}(a) = 1 - C^{\mathcal{I}}(a)$
conjunction	$(C \sqcap D)^{\mathcal{I}}(a) = \min(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a))$
disjunction	$(C \sqcup D)^{\mathcal{I}}(a) = \max(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a))$
exists restriction	$(\exists R.C)^{\mathcal{I}}(a) = \sup_{b \in \Delta^{\mathcal{I}}} \{\min(R^{\mathcal{I}}(a,b), C^{\mathcal{I}}(b))\}$
value restriction	$(\forall R.C)^{\mathcal{I}}(a) = \inf_{b \in \Delta^{\mathcal{I}}} \{\max(1 - R^{\mathcal{I}}(a,b), C^{\mathcal{I}}(b))\}$
at-most QCR	$(\leq pS.C)^{\mathcal{I}}(a) = \inf\limits_{b_1,\ldots,b_{p+1} \in \Delta^{\mathcal{I}}} \max\limits_{i=1}^{p+1} \{\max(1 - S^{\mathcal{I}}(a,b_i), 1 - C^{\mathcal{I}}(b_i))\}$
at-least QCR	$(\geq pS.C)^{\mathcal{I}}(a) = \sup\limits_{b_1,\ldots,b_p \in \Delta^{\mathcal{I}}} \min\limits_{i=1}^{p} \{\min(S^{\mathcal{I}}(a,b_i), C^{\mathcal{I}}(b_i))\}$
nominal	$\{o\}^{\mathcal{I}}(a) = 1$ if $a \in \{o^{\mathcal{I}}\}$, otherwise $\{o\}^{\mathcal{I}}(a) = 0$
inverse roles	$(R^-)^{\mathcal{I}}(b,a) = R^{\mathcal{I}}(a,b)$

Using the fuzzy set theoretic operations, fuzzy interpretations can be extended to interpret f_{KD}-$\mathcal{SHOIQ}$-concepts and roles. The semantics are depicted in Table 2. Note that, in the semantic of QCRs, all b_i are pair-wise different, i.e. $b_i \neq b_j$, $i \neq j$. These semantics are based on those presented in [52]. As it was later shown [50] under these semantics deciding the key inference problems in f_{KD}-DLs can be done effectively by tableaux-based algorithms using a form of *counting* (as in the crisp case). For other types of semantics the reader is referred to [41,20,10].

An f_{KD}-$\mathcal{SHOIQ}$ *TBox* $\mathcal{T}$ is a finite set of concept inclusion axioms of the form $C \sqsubseteq D$, for C, D f_{KD}-$\mathcal{SHOIQ}$-concpets, like in classical $\mathcal{SHOIQ}$. A fuzzy interpretation $\mathcal{I}$ satisfies $C \sqsubseteq D$ if $\forall a \in \Delta^{\mathcal{I}}, C^{\mathcal{I}}(a) \leq D^{\mathcal{I}}(a)$. If $\mathcal{I}$ satisfies each axiom in $\mathcal{T}$ then $\mathcal{I}$ is called a *model* of $\mathcal{T}$. Note that we do not use graded (fuzzy) subsumption axioms [52,5]. As it was shown in [5], fuzzy subsumption axioms in fuzzy DLs where implication is interpreted using the Kleene-Dienes fuzzy implication can lead to counter-intuitive semantics.

An f_{KD}-$\mathcal{SHOIQ}$ *RBox* $\mathcal{R}$ is a finite set of role axioms of the form $\mathsf{Trans}(R)$, called *transitive role axioms*, or of the form $R \sqsubseteq S$, called role inclusion axioms, where R, S are f_{KD}-$\mathcal{SHOIQ}$-roles. A fuzzy interpretation $\mathcal{I}$ satisfies $\mathsf{Trans}(R)$ if for every $a, c \in \Delta^{\mathcal{I}}$ we have $R^{\mathcal{I}}(a,c) \geq \sup_{b \in \Delta^{\mathcal{I}}} \{\min(R^{\mathcal{I}}(a,b), R^{\mathcal{I}}(b,c))\}$, while it satisfies $R \sqsubseteq S$ if $\forall a, b \in \Delta^{\mathcal{I}}$, $R^{\mathcal{I}}(a,b) \leq S^{\mathcal{I}}(a,b)$. If a fuzzy interpretation $\mathcal{I}$ satisfies each axiom in $\mathcal{R}$, then it is called a *model* of $\mathcal{R}$.

An f_{KD}-$\mathcal{SHOIQ}$ *ABox* $\mathcal{A}$ is a finite set of fuzzy assertions and (in)equality axioms. For $a, b \in \mathbf{I}$, a *fuzzy assertion* [51] is of the form $(a : C) \bowtie n$, $((a,b) : R) \bowtie n$, or $((a,b) : \neg R) \bowtie n$ where $\bowtie \in \{\geq, >, \leq, <\}$ and $n \in (0,1]$ if $\bowtie \in \{\geq, <\}$, while $n \in [0,1)$ if $\bowtie \in \{\leq, >\}$. We call assertions defined by $\geq, >$, *positive* assertions, while those defined by $\leq, <$ *negative* assertions. An *(in)equality axiom* is of the form $a \doteq b$, or $a \not\doteq b$. A fuzzy interpretation $\mathcal{I}$ satisfies $(a : C) \geq n$ $((a,b) : R \geq n)$ if $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq n$ $(R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq n)$, it satisfies $a \doteq b$ if $a^{\mathcal{I}} = b^{\mathcal{I}}$ and it satisfies $a \not\doteq b$ if $a^{\mathcal{I}} \doteq b^{\mathcal{I}}$; similarly with the other inequalities. A fuzzy interpretation satisfies a fuzzy *ABox* $\mathcal{A}$ if it satisfies all fuzzy assertions in $\mathcal{A}$. In this case, we say $\mathcal{I}$ is a *model* of $\mathcal{A}$. If $\mathcal{A}$ has a model then we say that it is *consistent*, otherwise it is *inconsistent*. Finally, for an ABox $\mathcal{A}$ let $N^{\mathcal{A}}$ denote the following set of degrees:

$$N^{\mathcal{A}} = \{0, 0.5, 1\} \cup \{1 - n, n \mid (a : C \bowtie n) \in \mathcal{A} \text{ or } ((a,b) : R) \bowtie n \in \mathcal{A}\} \tag{1}$$

A fuzzy knowledge base Σ is a triple $\langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$, where $\mathcal{T}$ is an f_{KD}-$\mathcal{SHOIQ}$ TBox, $\mathcal{R}$ an f_{KD}-$\mathcal{SHOIQ}$ RBox and $\mathcal{A}$ an f_{KD}-$\mathcal{SHOIQ}$ ABox. If a fuzzy interpretation $\mathcal{I}$ is a model of $\mathcal{T}$, $\mathcal{R}$, and $\mathcal{A}$, then $\mathcal{I}$ is called a model of Σ. Note that models can be arbitrary. However, it has been shown in the literature [25] that in fuzzy DLs under the standard fuzzy operators, if Σ has a model, then Σ also has a so-called *witnessed model*, i.e., one where if $(\exists R.C)^{\mathcal{I}}(a) = n$ then there exists $b \in \Delta^{\mathcal{I}}$ (the witness) such that $\min(R^{\mathcal{I}}(a,b), C^{\mathcal{I}}(b)) = n$. Hence, in the following, when we speak of a 'model' we always assume a witnessed one.

In the rest of the paper we will make use of the following notation: Let $\rhd \in \{\geq, >\}$ and $\lhd \in \{\leq, <\}$. By $\neg\bowtie$ we denote the *negation* of an inequality—that is, $\neg \geq = <$ and $\neg < = \geq$ (the rest of the cases are defined in a similar way). Additionally, by $\bowtie^-$ we denote the *reflexion* of inequalities—that is, $\geq^- = \leq$, $<^- = >$ and so on. Finally, we use the symbol $+$ to denote the *strengthening* or *weakening* of an inequality—that is, $+ \geq = >$ (strengthens) and $+ < = \leq$ (weakens) (the rest of the cases are defined similarly). We also use notation like $+\rhd$ and $\rhd^-$; this is to be understood as follows: if $\rhd = \geq$, then $+\rhd = >$, while if $\rhd = >$, then $+\rhd = \geq$ (similarly for $\rhd^-$).

3.1.1 Syntactic Simplifications

Now we introduce several assumptions that do not affect the generality of our results and which would significantly simplify the presentation of our algorithms.

We can assume, without loss of generality, that $\mathcal{A}$ does not contain axioms of the form $a \doteq b$ or $c \not\doteq d$ as such axioms can be expressed using the assertions $a : \{b\} \geq 1$ and $c : \neg\{d\} \geq 1$, respectively.

Additionally, we can assume that no fuzzy assertions with inequality $<$ or $\leq$ exist. On the one hand, an assertion of the form $(a : C) \leq n$ can be transformed to the equivalent assertion $(a : \neg C) \geq 1 - n$. On the other hand, an assertion of the form $((a,b) : R) \leq n$ can be transformed into $a : \forall R.\neg\{b\} \geq 1 - n$; similarly for assertions that use $<$. Using this assumption we are able to reduce the number of tableaux expansion rules of the presented algorithm to half and considerably simplify the presentation.

Moreover, as it has been shown in the literature [35,48], we can also assume that no assertions with the inequality $>$ appear in an ABox $\mathcal{A}$. More precisely, an assertion of the form $(a : C) > n \in \mathcal{A}$ can be replaced by an assertion of the form $(a : C) \geq n + \varepsilon$, where ε is a small number converging to 0. An actual value for ε can be computed for $\mathcal{A}$ by the set $N^{\mathcal{A}}$ defined as in equation (1): first we order the values in $N^{\mathcal{A}}$ and then we take a fraction of the smallest difference $n_{i+1} - n_i$ for each $n_i, n_{i+1} \in N^{\mathcal{A}}$. It has been shown that the initial fuzzy ABox is consistent iff the normalised one is [35]; hence, in the following we only assume ABoxes with assertions of the form $(a : C) \geq n$ and $((a,b) : R) \geq n'$, where $n, n' \in (0,1]$.

Finally, we also assume that concepts are in their *negation normal form* (NNF) [27], i.e., negations occur only in front of concept names or nominals. An f$_{KD}$-$\mathcal{SHOIQ}$-concept can be transformed into an equivalent one in NNF by pushing negations inwards making use of the De Morgan laws, which are satisfied by the standard fuzzy operators, and the dualities between concepts $\top$ and $\bot$, between the operators $\exists$ and $\forall$, and between $\geq$ and $\leq$. Also concepts of the form $(a : \neg(\geq 0R.C)) \rhd n$ are replaced by, $(a : \bot) \geq 1$. For a fuzzy concept D, we use $\sim D$ to denote the NNF of $\neg D$.

3.1.2 Inference Problems and their Reduction

Now we define the inference problems of fuzzy DLs as well as how these are reduced to KB (un)satisfiability.

An f$_{KD}$-$\mathcal{SHOIQ}$-concept C is called *n-satisfiable* if there exists some fuzzy interpretation $\mathcal{I}$ and some $a \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a) = n$ and $n \in (0,1]$. A fuzzy knowledge base Σ is *satisfiable* if there exists a fuzzy interpretation $\mathcal{I}$ which satisfies all axioms in Σ. Furthermore, a concept C is *subsumed* by a concept D w.r.t. $\mathcal{T}$ if for all models $\mathcal{I}$ of $\mathcal{T}$ and all $a \in \Delta^{\mathcal{I}}$ we have $C^{\mathcal{I}}(a) \leq D^{\mathcal{I}}(a)$. Finally, for Φ a fuzzy assertion or a concept subsumption, Σ *entails* Φ, written $\Sigma \models \Phi$, iff any model of Σ also satisfies Φ.

As with classical DLs in fuzzy DLs inference problems can be reduced to to fuzzy knowledge base satisfiability [51,49]. This is important in order to devise one algorithm that can decide all problems. Let $\Sigma = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$ be a fuzzy KB (with $\mathcal{A}$ normalised). Then, C is n-satisfiable w.r.t. Σ iff $\langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{(a : C) \geq n\} \rangle$ is satisfiable. Moreover, for a classical $\mathcal{SHOIQ}$ assertion $a : C$, we have $\Sigma \models (a : C) \geq n$ iff $\langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{(a : \neg C) \geq n + \varepsilon\} \rangle$ is unsatisfiable. Furthermore, for C and D two f$_{KD}$-$\mathcal{SHOIQ}$-concepts $\Sigma \models C \sqsubseteq D$ iff $\Sigma_n = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{(a : C) \geq n, (a : \neg D) \geq 1 - n + \varepsilon\} \rangle$ for every $n \in (0,1]$. Clearly the last reduction is not suitable for a practical implementation. As shown by Straccia, for f$_{KD}$-$\mathcal{ALC}$ KBs it suffices to check the unsatisfiability of Σ_n only for two arbitrarily selected degrees $n \in (0, 0.5]$ and $n \in (0.5, 1]$. Stoilos et al. [49] later observed that for f$_{KD}$-$\mathcal{SHOIQ}$ KBs we need to check for additional degrees since in the presence of nominals the ABox (and hence, the degrees in its fuzzy assertions) can interact with $(a : C) \geq n$ and $(a : \neg D) \geq 1 - n + \varepsilon$. More precisely, we need to check for each $n \in N^{\mathcal{A}}$ where $N^{\mathcal{A}}$ is as defined in equation (1). Note that for the latter approach to work it is essential that $\mathcal{A}$ is normalised. This is not observed in [49].

3.2 The Fuzzy Nominal Constructor: f$_{KD}$-$\mathcal{SHO}_{\mathtt{f}}\mathcal{IQ}$

Bobillo et al. [5] proposed a fuzzy extension of the nominal concept constructor called *fuzzy nominals*. Assume that we want to describe the concept of the German speaking countries as the set that contains the element '*germany*', to a degree 1, '*austria*' also to a degree 1 and '*switzerland*' to a degree 0.67 since only 67% of the population of Switzerland speak German. This is not possible using the constructors of the fuzzy DL f$_{KD}$-$\mathcal{SHOIQ}$. More precisely, an axiom of the form $\mathsf{GermanSpeaking} \equiv \{germany\} \sqcup \{austria\} \sqcup \{switzerland\}$ would imply that $switzerland^{\mathcal{I}}$ belongs to $\mathsf{GermanSpeaking}^{\mathcal{I}}$ to a degree 1; similarly, the fuzzy assertions $(germany : \mathsf{GermanSpeaking}) \geq 1$, $(austria : \mathsf{GermanSpeaking}) \geq 1$, and $(switzerland : \mathsf{GermanSpeaking}) \geq 0.67$ do not give the intended semantics as then we cannot guarantee that only these are the members of the concept GermanSpeaking. However, this is easily expressible using fuzzy nominals using the following axiom:

$$\mathsf{GermanSpeaking} \equiv \{germ, 1\} \sqcup \{aus, 1\} \sqcup \{switz, 0.67\}.$$

where the degrees next to the individuals denote the degree to which the objects belong in the set $\mathsf{GermanSpeaking}^{\mathcal{I}}$.

Formally, let $o \in \mathbf{I}$ be an individual and let $n \in (0,1]$ be a degree. Then, $\{o,n\}$ is an f$_{KD}$-$\mathcal{SHO}_{\mathtt{f}}\mathcal{IQ}$-concept. The semantics of the fuzzy nominal constructor is given by the following equation:

$$\{o,n\}^{\mathcal{I}}(a) = \begin{cases} n, \; a = o^{\mathcal{I}} \\ 0, \text{ otherwise} \end{cases}$$

In the following, in order to distinguish the fuzzy DL that uses nominals with the one that does not we call the former language f$_{KD}$-$\mathcal{SHO}_{\mathtt{f}}\mathcal{IQ}$.

As shown in [49], in the presence of fuzzy nominals the reduction of concept subsumption to KB satisfiability needs to be revised. More precisely, $\Sigma \models C \sqsubseteq D$ if and only if

$\langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{(a : C) \geq n, (a : \neg D) \geq 1 - n + \varepsilon\}\rangle$ is unsatisfiable for each n in the set:

$$X^{\Sigma} = N^{\Sigma} \cup \{n_i \mid \{o_i, n_i\} \text{ appears in } C\} \cup \{n_i + \varepsilon \mid \{o_i, n_i\} \text{ appears in } D\}$$

where $N^{\Sigma} = N^{\mathcal{A}} \cup \{n_i \mid \{o_i, n_i\} \in \Sigma\}$ and ε is again a small number converging to 0.

4 Reasoning in f$_{KD}$-$\mathcal{SHOIQ}$

In Section 3 we have shown how all inference problems can be reduced to the problem of fuzzy knowledge base satisfiability. To check satisfiability of a fuzzy KB we use tableaux algorithms for expressive fuzzy DLs, like those presented for f$_{KD}$-$\mathcal{SI}$ and f$_{KD}$-$\mathcal{SHIN}$ [45], f$_{KD}$-$\mathcal{ALCIQ}$ [45], and for general TBoxes [48], but extended appropriately to cover the new constructors of f$_{KD}$-$\mathcal{SHOIQ}$. Tableaux algorithms are model-constructing calculi [24] that given a knowledge base they try to construct a finite structure which consists of an abstraction of a model of the KB [29].

First, we define the notions of subconcepts of a concept D and subconcepts of a knowledge base and then proceed with the definition of a *tableau*, which gives a characterisation of models of f$_{KD}$-$\mathcal{SHOIQ}$ KBs.

Definition 1 Let D be an f$_{KD}$-$\mathcal{SHOIQ}$-concept. The set of *subconcepts* of D, denoted by $sub(D)$, is inductively defined as follows:

$$\begin{aligned}
sub(A) &= \{A\} \text{ for every atomic concept } A \in \mathbf{C},\\
sub(\{o\}) &= \{\{o\}\},\\
sub(C \sqcap D) &= \{C \sqcap D\} \cup sub(C) \cup sub(D),\\
sub(C \sqcup D) &= \{C \sqcup D\} \cup sub(C) \cup sub(D),\\
sub(\exists R.C) &= \{\exists R.C\} \cup sub(C),\\
sub(\forall R.C) &= \{\forall R.C\} \cup sub(C),\\
sub(\geq pR.C) &= \{\geq pR.C\} \cup sub(C), \text{ and}\\
sub(\leq pR.C) &= \{\leq pR.C\} \cup sub(C)
\end{aligned}$$

Let additionally $\mathcal{R}$ be an RBox. Then, $cl(D, \mathcal{R})$ is the smallest set of f$_{KD}$-$\mathcal{SHOIQ}$-concepts that satisfies the following:

- $D \in cl(D, \mathcal{R})$,
- $cl(D, \mathcal{R})$ is closed under subconcepts and application of $\sim$, and
- if $\forall R.C \in cl(D, \mathcal{R})$, $P \,\underline{\circledast}\, R$ and $\mathsf{Trans}(P)$, then $\forall P.C \in cl(D, \mathcal{R})$

Finally, let $cl(\Sigma) = \bigcup_{(a:D)\rhd n \in \mathcal{A}} cl(D, \mathcal{R}) \bigcup_{C \sqsubseteq D \in \mathcal{T}} cl(C, \mathcal{R}) \cup cl(D, \mathcal{R})$. ◇

Definition 2 Let $\Sigma = \langle \mathcal{T}, \mathcal{R}, \mathcal{A}\rangle$ be an f$_{KD}$-$\mathcal{SHOIQ}$ KB, let $\mathbf{R}_{\Sigma}$ be the set of roles occurring in Σ together with their inverses, and let $\mathbf{I}_{\Sigma}$ be the set of individuals appearing in Σ (either in assertions or in nominal concepts); then, a *fuzzy tableau* T for Σ is defined to be a quadruple ($\mathbf{S}$, $\mathcal{L}$, $\mathcal{E}$, $\mathcal{V}$) such that, $\mathbf{S}$ is a set of elements, $\mathcal{L} : \mathbf{S} \times cl(\Sigma) \to [0,1]$ is a function that maps each pair of elements of $\mathbf{S}$ and $cl(\Sigma)$ to a degree, $\mathcal{E} : \mathbf{R}_{\Sigma} \times \mathbf{S} \times \mathbf{S} \to [0,1]$ is a function that maps each role of $\mathbf{R}_{\Sigma}$ and pair of elements to a degree, and $\mathcal{V} : \mathbf{I}_{\Sigma} \to \mathbf{S}$ maps individuals occurring in Σ to elements of $\mathbf{S}$.

Let $\sharp$ denote the cardinality of a set and for a role R and tableau T let also the function R^T defined as follows: $R^T(s, \rhd, n, C) = \{t \in \mathbf{S} \mid \mathcal{E}(R, \langle s, t\rangle) \rhd n \text{ and } \mathcal{L}(s, C) \rhd n\}$. For all $s, t \in \mathbf{S}$, $C, E \in cl(\Sigma)$, $n \in N^{\mathcal{A}}$ and $R \in \mathbf{R}_{\Sigma}$, a fuzzy tableau T satisfies the following conditions:

1. $\mathcal{L}(s,\bot) = 0$ for all $s \in \mathbf{S}$,
2. $\mathcal{L}(s,\top) = 1$ for all $s \in \mathbf{S}$,
3. If $\mathcal{L}(s,\neg C) \rhd n$, then we don't have $\mathcal{L}(s,C) + \rhd 1-n$,
4. If $\mathcal{L}(s,C \sqcap E) \rhd n$, then $\mathcal{L}(s,C) \rhd n$ and $\mathcal{L}(s,E) \rhd n$,
5. If $\mathcal{L}(s,C \sqcup E) \rhd n$, then $\mathcal{L}(s,C) \rhd n$ or $\mathcal{L}(s,E) \rhd n$,
6. If $\mathcal{L}(s,\forall R.C) \rhd n$, then either $\mathcal{E}(R,\langle s,t\rangle) \rhd^{-} 1-n$ or $\mathcal{L}(t,C) \rhd n$,
7. If $\mathcal{L}(s,\exists R.C) \rhd n$, then there exists $t \in \mathbf{S}$ such that $\mathcal{E}(R,\langle s,t\rangle) \rhd n$ and $\mathcal{L}(t,C) \rhd n$,
8. If $\mathcal{L}(s,\forall R.C) \rhd n$ and for some $P \,\underline{\circledast}\, R$ we have $\mathsf{Trans}(P)$, then either $\mathcal{E}(P,\langle s,t\rangle) \rhd^{-} 1-n$, or $\mathcal{L}(t,\forall P.C) \rhd n$,
9. $\mathcal{E}(R,\langle s,t\rangle) \rhd n$ iff $\mathcal{E}(\mathsf{Inv}(R),\langle t,s\rangle) \rhd n$,
10. If $\mathcal{E}(R,\langle s,t\rangle) \rhd n$ and $R \,\underline{\circledast}\, S$, then $\mathcal{E}(S,\langle s,t\rangle) \rhd n$,
11. If $\mathcal{L}(s,\geq pR.C) \rhd n$, then $\sharp R^T(s,\rhd,n,C) \geq p$,
12. If $\mathcal{L}(s,\leq pR.C) \rhd n$, then $\sharp R^T(s,+\rhd,1-n,C) \leq p$,
13. If $\mathcal{L}(s,\leq pR.C) \rhd n$ and $\mathcal{E}(R,\langle s,t\rangle) + \rhd 1-n$, then either $\mathcal{L}(t,\neg C) \rhd n$ or $\mathcal{L}(t,C) + \rhd 1-n$,
14. If $\mathcal{L}(x,\{o\}) = 1$ and $\mathcal{L}(y,\{o\}) = 1$, for some $o \in \mathbf{I}_\Sigma$, then $x = y$,
15. If $\mathcal{L}(x,\{o\}) \rhd n$, then $\mathcal{L}(x,\{o\}) = 1$,
16. If $C \sqsubseteq D \in \mathcal{T}$, then either $\mathcal{L}(s,\neg C) > 1-n$ or $\mathcal{L}(s,D) \geq n$, for all $s \in \mathbf{S}$ and $n \in N^{\mathcal{A}}$,
17. If $(a : C) \geq n \in \mathcal{A}$, then $\mathcal{L}(\mathcal{V}(a),C) \geq n$,
18. If $((a,b) : R) \geq n \in \mathcal{A}$, then $\mathcal{E}(R,\langle \mathcal{V}(a),\mathcal{V}(b)\rangle) \geq n$,
19. If $(a : C) \geq n \in \mathcal{A}$, $(((a,b) : R) \geq n \in \mathcal{A})$, then $\mathcal{L}(\mathcal{V}(a),\{a\}) = 1$ $(\mathcal{L}(\mathcal{V}(a),\{a\}) = 1$ and $\mathcal{L}(\mathcal{V}(b),\{b\}) = 1)$

◇

The conditions that a fuzzy tableau needs to satisfy are motivated by the semantics of fuzzy interpretations and the properties of the fuzzy operators that are used by the f$_{KD}$-$\mathcal{SHOIQ}$ language. For example, Property 4 dictates that if some object s belongs to $C \sqcap E$ to a degree greater or equal (strictly greater) than n, then s must belong to both C and E to a degree greater or equal (strictly greater) than n. This is because, by the semantics, $(C \sqcap D)^{\mathcal{I}}(s) \rhd n$ implies $\min(C^{\mathcal{I}}(s),E^{\mathcal{I}}(s)) \rhd n$, hence we must have both $C^{\mathcal{I}}(s) \rhd n$ and $E^{\mathcal{I}}(s) \rhd n$. Similarly, Property 12 is a result of the semantics of cardinality restrictions analysed in [50]; if $\mathcal{L}(s,\leq pR.C) \geq n$ then there are at-most p elements $t \in \mathbf{S}$ such that $\mathcal{E}(R,\langle s,t\rangle) > 1-n$ and $\mathcal{L}(t,C) > 1-n$ $(+\geq \equiv >)$, and if $\mathcal{L}(s,\leq pR.C) > n$, then there are at-most p elements $t \in \mathbf{S}$ such that $\mathcal{E}(R,\langle s,t\rangle) \geq 1-n$ and $\mathcal{L}(t,C) \geq 1-n$ (since $+ > \equiv \geq$) as it was shown in [50]. Additionally, Property 16 is required in order to faithfully capture the semantics of the axioms in the TBox [48]. This property is based on the observation that a fuzzy interpretation $\mathcal{I}$ satisfies $C \sqsubseteq D \in \mathcal{T}$ iff it satisfies either $(a : C)^{\mathcal{I}} < n$ (i.e. $(a : \neg C)^{\mathcal{I}} > 1-n$), or $(a : D)^{\mathcal{I}} \geq n$ for every $n \in (0,1]$. However, as shown in [35], one can restrict n to vary over the finite set $N^{\mathcal{A}}$.

The next lemma, which proof is given in the appendix, shows the desirable connection between the existence of a model for an f$_{KD}$-$\mathcal{SHOIQ}$ KB and a fuzzy tableau.

Lemma 1 *An f$_{KD}$-$\mathcal{SHOIQ}$ knowledge base Σ is satisfiable iff there exists a fuzzy tableau T for Σ.*

4.1 A Tableaux Algorithm for f$_{KD}$-$\mathcal{SHOIQ}$

In order to decide knowledge base satisfiability a procedure that constructs a fuzzy tableau for an f$_{KD}$-$\mathcal{SHOIQ}$ KB has to be devised. Such a procedure will be based on tableaux algorithms. In the current section we will provide the technical details of the tableaux algorithm for f$_{KD}$-$\mathcal{SHOIQ}$.

Like the algorithm presented in [30], our algorithm works on a *completion-graph* **G**—that is, a graph whose vertices correspond to individuals labelled with concepts and degrees of memberships and whose edges correspond to relations between individuals, also labelled with degrees of membership. Moreover, due to the presence of transitive roles the termination of the algorithm is ensured by the use of *blocking*, which stops the algorithm after a specific type of cycle has been detected in the expansion process. Additionally, as f$_{KD}$-$\mathcal{SHOIQ}$ provides inverse roles, transitive role axioms and qualified number restrictions it lacks the finite-model property, that is, there are f$_{KD}$-$\mathcal{SHOIQ}$-concepts that are satisfiable w.r.t. a knowledge base only in infinite interpretations and thus only an infinite tableau (model) exists for them. One such concept is the following [29]:

$$F \equiv \neg C \sqcap \exists P^-.(C \sqcap \leq 1P) \sqcap \forall S^-.(\exists P^-.(C \sqcap \leq 1P))$$

with $\mathcal{R} = \{\mathsf{Trans}(S), P \sqsubseteq S\}$. Hence, in order to construct a correct tableau out of the possibly blocked graph one needs to repeatedly copy (*unravel*) the sub-graph underneath the node that causes blocking. The appropriate blocking technique which allows such unravelling is called *pair-wise* blocking [29].

Furthermore, since f$_{KD}$-$\mathcal{SHOIQ}$ provides also nominals we have to ensure that the unravelling process does not violate possible number restrictions on roles that connect nodes of the completion-graph with nominals. This problem was illustrated in [30] using the following example:

Example 1 Let Σ_c be a knowledge base that contains the following set of axioms:

$$\top \sqsubseteq \exists R^-.\{o\}$$
$$\{o\} \sqsubseteq \leq 3R.\top$$

and let also the fuzzy assertion $(a : F) \geq 0.6 \in \mathcal{A}$, where F is the f$_{KD}$-$\mathcal{SHOIQ}$-concept defined previously. The first axiom implies that every object of $\Delta^{\mathcal{I}}$ is connected with $o^{\mathcal{I}}$ through role $(R^-)^{\mathcal{I}}$. The second axiom specifies that there exist at-most 3 objects in $\Delta^{\mathcal{I}}$ connected to $o^{\mathcal{I}}$ with this role. As we noted before F is satisfiable only into an infinite interpretation, consequently the above KB is unsatisfiable (since infinite number of objects implies infinite number of roles $(R^-)^{\mathcal{I}}$ with $o^{\mathcal{I}}$, but only 3 are allowed). Nevertheless, in order for the tableaux algorithm to correctly identify this unsatisfiability it should not apply blocking before enough individuals have been introduced. ⊣

In order to correctly reason with such knowledge bases, Horrocks and Sattler [30] propoed the *NN*-rule which forces a form of locality when number restrictions are related with nominals. In the following we will extend the *NN*-rule to address this problem also in f$_{KD}$-$\mathcal{SHOIQ}$. In Example 2 we will show how this rule can detect the unsatisfiability of the above knowledge base.

Definition 3 Let Σ be an f$_{KD}$-$\mathcal{SHOIQ}$ knowledge base, let $\mathbf{R}_\Sigma$ be the set of roles appearing in Σ and let **I** be a set of individuals. A *completion-graph* **G** for an f$_{KD}$-$\mathcal{SHOIQ}$ knowledge base Σ is a directed graph $\mathbf{G} = (\mathbf{V}, E, \mathcal{L}, \dot{\neq}, \dot{=})$, where each node $x \in V$ is labelled with a set of pairs $\langle C, n\rangle \in \mathcal{L}(x)$ such that

$$C \in cl(\Sigma) \cup \{\{a\} \mid a \in \mathbf{I}\} \cup \{\leq p'R.C \mid (\leq pR.C) \in cl(\Sigma) \text{ and } p' \leq p\}$$

called *concept pairs*, and each edge $\langle x,y\rangle \in E$ is labelled with a set of role names $\mathcal{L}(\langle x,y\rangle) = \{\langle R,n\rangle\}$, where $R \in \mathbf{R}_\Sigma$ are possibly inverse roles occurring in Σ, called *role pairs*.

If nodes x and y are connected by an edge $\langle x,y\rangle$ with $\langle P,n\rangle \in \mathcal{L}(\langle x,y\rangle)$, and $P \sqsubseteq^* R$, then y is called an R_n*-successor* of x and x is called an R_n*-predecessor* of y. If y is an R_n-successor or an $\mathsf{Inv}(R)_n$*-predecessor* of x, then y is called an R_n-neighbour of x. As usual, *ancestor* is the transitive closure of *predecessor*.

For y an R_n-neighbour of x, we say that the edge $\langle x,y\rangle$ *conjugates* with the pair $\langle\neg R,m\rangle$[4] if $n+m>1$. The notion of conjugation can be straightforwardly extended to concept pairs.

For two roles P, R, a concept C, a node x in $\mathbf{G}$ and a membership degree n we define: $R_c^{\mathbf{G}}(x,n,C) = \{y \mid y$ is an $R_{n'}$-neighbour of x, $\langle x,y\rangle$ is conjugated with $\langle\neg R,n\rangle$, and $\mathcal{L}(y) \cup \{\langle\neg C,n\rangle\}$ contains a conjugation$\}$.

A node y is called *blockable* if it contains no nominals in its label $\mathcal{L}(y)$, otherwise it is called *nominal*. A node x is *label blocked* iff it has ancestors x', y and y' such that

1. x is a successor[5] of x' and y a successor of y',
2. $\mathcal{L}(y)$ does not contain a pair with a nominal,
3. y,x and all nodes on the path from y to x are blockable,
4. $\mathcal{L}(x) = \mathcal{L}(y)$ and $\mathcal{L}(x') = \mathcal{L}(y')$ and,
5. $\mathcal{L}(\langle x',x\rangle) = \mathcal{L}(\langle y',y\rangle)$.

In this case we say that y blocks x. A node y is indirectly blocked iff one of its ancestors is blocked.

For a node x, $\mathcal{L}(x)$ is said to contain a clash if it contains one of the following:

- two conjugated pairs,
- a pair $\langle\bot,n\rangle$, with $n>0$, or
- some pair $\langle\leq pR.C,n\rangle$ and x has $p+1$ R_{n_i}-neighbours $y_0,\ldots,y_p$, all $\langle x,y_i\rangle$ are conjugated with $\langle\neg R,n\rangle$, $\mathcal{L}(y_i)\cup\{\langle\neg C,n\rangle\}$ contains a clash and $y_i \neq y_j$, for all $0 \leq i < j \leq p$, or
- for some $o \in \mathbf{I}$, there are $x \neq y$ with $\langle\{o\},1\rangle \in \mathcal{L}(x) \cap \mathcal{L}(y)$.

◇

Intuitively, a completion-graph encodes information about the membership of individuals in concepts as well as the various fuzzy relations between pairs of individuals. For example, $\langle C,n\rangle \in \mathcal{L}(x)$ implies that x belongs to C to a degree greater or equal than n. Additionally, $\not\doteq$ ($\doteq$) keep track of the inequalities (equalities) between nodes of $\mathbf{G}$.

We will now define the tableaux algorithm for f$_{KD}$-$\mathcal{SHOIQ}$ knowledge bases.

Definition 4 Let $\Sigma = \langle\mathcal{T},\mathcal{R},\mathcal{A}\rangle$ be an f$_{KD}$-$\mathcal{SHOIQ}$ knowledge base. A *tableaux algorithm* initialises a graph $\mathbf{G}$ to contain:

1. a root node x_{a_i}, for each individual $a_i \in \mathbf{I}_\Sigma$ appearing in the KB Σ, labelled with $\mathcal{L}(x_{a_i})$ such that for each $(a_i : C_i) \geq n \in \mathcal{A}$ we have $\{\langle\{a_i\},1\rangle,\langle C,n\rangle\} \subseteq \mathcal{L}(x_{a_i})$,
2. an edge $\langle x_{a_i},x_{a_j}\rangle$, for each $((a_i,a_j) : S_i) \geq n \in \mathcal{A}$, labelled with a set $\mathcal{L}(\langle x_{a_i},x_{a_j}\rangle)$ such that for each $((a_i,a_j) : S_i) \geq n \in \mathcal{A}$ we have $\langle S_i,n\rangle \in \mathcal{L}(\langle x_{a_i},x_{a_j}\rangle)$,
3. the relations $\not\doteq$ and $\doteq$ to be empty.

[4] Note that this is an artificial pair used to check if a certain condition in $\mathcal{L}(\langle x,y\rangle)$ holds.

[5] A node x_2 is a successor of a node x_1 if $\langle x_1,x_2\rangle \in E$.

G is then expanded by repeatedly applying the tableaux rules from Table 3. The graph is *complete* when, for some node x, $\mathcal{L}(x)$ contains a clash, or none of the tableaux rules is applicable. The algorithm stops when a clash occurs; it answers 'Σ is satisfiable' iff the tableaux rules can be applied in such a way that they yield a complete and clash-free completion-graph, and 'Σ is satisfiable' otherwise.

The algorithm additionally uses the following functions.

Merging: Merging a node y into a node x, means that we add $\mathcal{L}(y)$ to $\mathcal{L}(x)$, "move" all the edges leading to y so that they lead to x and "move" all the edges leading from y to nominal nodes so that they lead from x to the same nominal nodes; then if y is not a root node we remove y (and blockable sub-trees below y) from the completion-graph, otherwise we set $\mathcal{L}(y)$ to the empty set and assert that $x \doteq y$. More precisely, merging a node y into a node x (written $\mathsf{Merge}(y,x)$) in $\mathbf{G} = (V, E, \mathcal{L}, \neq, \doteq)$ yields a graph that is obtained from **G** as follows:

1. For all nodes z such that $\langle z,y\rangle \in E$
   - If $\{\langle x,z\rangle, \langle z,x\rangle\} \cap E = \emptyset$, then add $\langle z,x\rangle$ to E and set $\mathcal{L}(\langle z,x\rangle) = \mathcal{L}(\langle z,y\rangle)$,
   - if $\langle z,x\rangle \in E$, then set $\mathcal{L}(\langle z,x\rangle) = \mathcal{L}(\langle z,x\rangle) \cup \mathcal{L}(\langle z,y\rangle)$,
   - if $\langle x,z\rangle \in E$, then set $\mathcal{L}(\langle x,z\rangle) = \mathcal{L}(\langle x,z\rangle) \cup \{\langle \mathsf{Inv}(R), n\rangle \mid \langle R,n\rangle \in \mathcal{L}(\langle z,y\rangle)\}$ and
   - remove $\langle z,y\rangle$ from E;
2. For all nominal nodes z such that $\langle y,z\rangle \in E$
   - If $\{\langle x,z\rangle, \langle z,x\rangle\} \cap E = \emptyset$, then add $\langle x,z\rangle$ to E and set $\mathcal{L}(\langle x,z\rangle) = \mathcal{L}(\langle y,z\rangle)$,
   - if $\langle x,z\rangle \in E$, then set $\mathcal{L}(\langle x,z\rangle) = \mathcal{L}(\langle x,z\rangle) \cup \mathcal{L}(\langle y,z\rangle)$,
   - if $\langle z,x\rangle \in E$, then set $\mathcal{L}(\langle z,x\rangle) = \mathcal{L}(\langle z,x\rangle) \cup \{\langle \mathsf{Inv}(R), n\rangle \mid \langle R,n\rangle \in \mathcal{L}(\langle y,z\rangle)\}$ and
   - remove $\langle y,z\rangle$ from E;
3. set $\mathcal{L}(x) = \mathcal{L}(x) \cup \mathcal{L}(y)$;
4. add $x \neq z$ for all z s.t. $y \neq z$;
5. if y is a root node then set $x \doteq y$; and
6. $\mathsf{Prune}(y)$.

Prune: Pruning a node y from a completion-graph **G**, yields a new graph that is obtained from **G** as follows:

1. for all successors z of y, remove $\langle y,z\rangle$ from **G**, and if z is blockable $\mathsf{Prune}(z)$;
2. if y is a root node set $\mathcal{L}(y) = \emptyset$, otherwise remove y.

Strategy of Rule Application: As noted in [30] in order to ensure termination, and in particular to fix an upper bound on the number of application of the *NN*-rule, the expansion rules must be applied according to the following strategy:

1. the $\{o\}_2$-rule is applied with highest priority,
2. then, the $\{o\}_1$-rule is applied,
3. next, the $\leq$ and *NN*-rules are applied, and they are applied first to nominal nodes with lower levels. In case they are both applicable to the same node, the *NN*-rule is applied first.
4. all other rules are applied with a lower priority. ♢

There are several remarks regarding the above definition. First, it is important to note that due to initialisation root nodes represent individuals that exist in some fuzzy assertion in the ABox and additionally are nominal nodes. Second, our definitions of the methods Merge and Prune slightly differ from the ones presented in [30]. More precisely, when merging y into x

Table 3: Tableaux rules for f_{KD}-$\mathcal{SHOIQ}$

Rule		**Description**
$\sqcap$	if 1.	$\langle C_1 \sqcap C_2, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked, and
	2.	$\{\langle C_1, n\rangle, \langle C_2, n\rangle\} \not\subseteq \mathcal{L}(x)$
	then	$\mathcal{L}(x) := \mathcal{L}(x) \cup \{\langle C_1, n\rangle, \langle C_2, n\rangle\}$
$\sqcup$	if 1.	$\langle C_1 \sqcup C_2, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked, and
	2.	$\{\langle C_1, n\rangle, \langle C_2, n\rangle\} \cap \mathcal{L}(x) = \emptyset$
	then	$\mathcal{L}(x) := \mathcal{L}(x) \cup \{C\}$ for some $C \in \{\langle C_1, n\rangle, \langle C_2, n\rangle\}$
$\exists$	if 1.	$\langle \exists R.C, n\rangle \in \mathcal{L}(x)$, x is not blocked,
	2.	x has no R_n-neighbour y with $\langle C, n\rangle \in \mathcal{L}(y)$
	then	create a new node y and set $\mathcal{L}(\langle x, y\rangle) := \{\langle R, n\rangle\}$, $\mathcal{L}(y) := \{\langle C, n\rangle\}$
$\forall$	if 1.	$\langle \forall R.C, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked, and
	2.	x has an R_{n_1}-neighbour y with $\langle C, n\rangle \notin \mathcal{L}(y)$ and
	3.	$\langle x, y\rangle$ conjugates with $\langle \neg R, n\rangle$
	then	$\mathcal{L}(y) := \mathcal{L}(y) \cup \{\langle C, n\rangle\}$
$\forall_+$	if 1.	$\langle \forall S.C, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked, and
	2.	there is some R, with $\mathsf{Trans}(R)$, and $R \sqsubseteq^* S$,
	3.	x has a R_{n_1}-neighbour y with $\langle \forall R.C, n\rangle \notin \mathcal{L}(y)$, and
	4.	$\langle x, y\rangle$ conjugates with $\langle \neg R, n\rangle$
	then	$\mathcal{L}(y) := \mathcal{L}(y) \cup \{\langle \forall R.C, n\rangle\}$
choose	if 1.	$\langle \leq pR.C, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked
	2.	there is an R-neighbour y of x, with $\{\langle \neg C, n\rangle, \langle C, 1-n+\varepsilon\rangle\} \cap \mathcal{L}(x) = \emptyset$, and
	3.	$\langle x, y\rangle$ conjugates with $\langle \neg R, n\rangle$
	then	$\mathcal{L}(y) := \mathcal{L}(y) \cup \{E\}$, for some $E \in \{\langle \neg C, n\rangle, \langle C, 1-n+\varepsilon\rangle\}$
$\geq$	if 1.	$\langle \geq pR.C, n\rangle \in \mathcal{L}(x)$, x is not blocked,
	2.	there are no p $R_{\triangleright n}$-neighbours $y_1, \ldots, y_p$ of x with $\langle C, n\rangle \in \mathcal{L}(y_i)$ with
		$y_i \neq y_j$ for $1 \leq i < j \leq p$
	then	create p new nodes $y_1, \ldots, y_p$, with $\mathcal{L}(\langle x, y_i\rangle) = \{\langle R, n\rangle\}$,
		$\mathcal{L}(y_i) := \{\langle C, n\rangle\}$ and $y_i \neq y_j$ for $1 \leq i < j \leq p$
$\leq$	if 1.	$\langle \leq pR.C, n\rangle \in \mathcal{L}(z)$, z is not indirectly blocked,
		$\sharp R_c^{\mathbf{G}}(z, n, C) > p$, there are two of them x, y, with no $y \not\doteq x$
	then	1. if x is a nominal or root node, then $\mathsf{Merge}(y, x)$
		2. else if y is a nominal or a root node or an ancestor of x, then $\mathsf{Merge}(x, y)$
		3. else $\mathsf{Merge}(y, x)$
$\{o\}_1$	if	for some $o \in \mathbf{I}$ there are two nodes x, y with $\langle \{o\}, 1\rangle \in \mathcal{L}(x) \cap \mathcal{L}(y)$ and not $x \not\doteq y$
	then	$\mathsf{Merge}(x, y)$
NN	if 1.	$\langle \leq pR.C, n\rangle \in \mathcal{L}(x)$, x is a nominal node, and there is a blockable
		node y of x in the set $R_c^{\mathbf{G}}(x, n, C)$ and x is a successor of y,
	2.	there is no m such that $1 \leq m \leq p$, $\langle \leq mR.C, n\rangle \in \mathcal{L}(x)$
		and $\sharp R_c^{\mathbf{G}}(x, n, C) \geq m$, m are nominal nodes $z_1, ..., z_m$ of x,
		with $z_i \not\doteq z_j$, for all $1 \leq i < j \leq m$
	then	1. guess m, with $1 \leq m \leq n$ and set $\mathcal{L}(x) := \mathcal{L}(x) \cup \{\langle \leq mR.C, n\rangle\}$
		2. create m new nodes $z_1, ..., z_m$, and set $\mathcal{L}(\langle x, z_i\rangle) := \{\langle R, 1-n+\varepsilon\rangle\}$,
		$\mathcal{L}(z_i) := \{\langle \{o_i\}, 1\rangle, \langle C, 1-n+\varepsilon\rangle\}$ for each $o_i \in \mathbf{I}$ new in $\mathbf{G}$, and
		$z_i \not\doteq z_j$ for $1 \leq i < j \leq m$
$\leq_{\{o\}}$	if. 1.	$\langle \leq pR.C, n\rangle \in \mathcal{L}(x)$, x is a nominal node, and there is a blockable
		node y of x in the set $R_c^{\mathbf{G}}(x, n, C)$
	2.	there exist m nominal $R_{1-n+\varepsilon}$-neighbours $z_1, \ldots, z_m$ of x
		with $\langle C, 1-n+\varepsilon\rangle \in \mathcal{L}(z_i)$ and $z_i \not\doteq z_j$ with $1 \leq i < j \leq m$ and
	3.	there is a nominal R-neighbour z of x contained in $R_c^{\mathbf{G}}(x, n, C)$ and not $z \not\doteq y$
	then	$\mathsf{Merge}(z, y)$
$\{o\}_2$	if 1.	$\langle \{o\}, n\rangle \in \mathcal{L}(x)$ ($\langle \neg\{o\}, n\rangle \in \mathcal{L}(x)$), and
	2.	$\langle \{o\}, 1\rangle \notin \mathcal{L}(x)$ ($\langle \neg\{o\}, 1\rangle \notin \mathcal{L}(x)$)
	then	$\mathcal{L}(x) := \mathcal{L}(x) \cup \{\langle \{o\}, 1\rangle\}$ ($\mathcal{L}(x) := \mathcal{L}(x) \cup \{\langle \neg\{o\}, 1\rangle\}$)
$\sqsubseteq$	if 1.	$C \sqsubseteq D \in \mathcal{T}$, x is not indirectly blocked, and
	2.	$\{\langle \neg C, 1-n+\varepsilon\rangle, \langle D, n\rangle\} \cap \mathcal{L}(x) = \emptyset$ for some $n \in N^{\mathcal{A}}$
	then	$\mathcal{L}(x) := \mathcal{L}(x) \cup \{E\}$ for some $E \in \{\langle \neg C, 1-n+\varepsilon\rangle, \langle D, n\rangle\}$

if y is a root node, then $(\mathcal{L}(y))$ is set to $\emptyset$ and we also have $x \doteq y$, whereas in [30] the node y is removed from the graph. The reason for this difference is that the algorithm in [30] deals with concept satisfiability, while here our algorithm directly works over ABox assertions. While in classical $\mathcal{SHOIQ}$ the presence of nominals allows one to 'absorb' an ABox into a single concept, this is not possible in f$_{KD}$-$\mathcal{SHOIQ}$ due to the fuzzy semantics. Third, although $\mathcal{A}$ is normalised (it only contains assertions that use the inequality $\geq$), the algorithm still needs to introduce assertions of the form $(a : C) > 1-n$ due to the semantics of some constructors (cf. Definition 2, Property 16). Hence, in order keep the pairs normalised and avoid the need to use triples $\langle C, \bowtie, n\rangle$ (as it has been done in previous works [50]) the algorithm introduces pairs of the form $\langle C, 1-n+\varepsilon\rangle$ for some ε. Note also that in this case, in order to identify a clash one does not necessarily need to compute an actual value for ε. If $\{\langle A,n\rangle, \langle \neg A, 1-n+\varepsilon\rangle\} \subseteq \mathcal{L}(x)$ then clearly $n+1-n+\varepsilon > 1$.

Example 2 We will apply our tableaux algorithm on the KB Σ_c defined in Section 4.1.

Firstly, the algorithm initialises a completion-graph $\mathbf{G}$ to contain the following nodes and node labels: $\mathcal{L}(x_a) = \{\langle F, 0.6\rangle, \langle\{a\}, 1\rangle\}$ and $\mathcal{L}(r_o) = \{\langle\{o\}, 1\rangle\}$. Moreover, the set of relative membership degrees is as follows: $N^{\mathcal{A}} = \{0, 0.4, 0.5, 0.6, 1\}$. Subsequently, the algorithm expands $\mathbf{G}$ by applying the tableaux rules from Table 3.

When the $\sqsubseteq$-rule is applied to the axiom $\top \sqsubseteq \exists R^-.\{o\}$ it either adds $\langle\neg\top, 1-n+\varepsilon\rangle$ or $\langle\exists R^-.\{o\}, n\rangle$ to $\mathcal{L}(x_a)$. Since this non-deterministic step is performed for every $n \in N^{\mathcal{A}}$ at some point the algorithm chooses $n = 0.6$. Then, clearly $\langle\neg\top, 0.6\rangle$ is a clash leaving only the choice $\langle\exists R^-.\{o\}, 0.6\rangle \in \mathcal{L}(x_a)$. Then, we have the following application of tableaux rules:

(1)	$\mathcal{L}(\langle x_a, x\rangle) := \{\langle R^-, 0.6\rangle\}, \mathcal{L}(x) := \{\langle\{o\}, 0.6\rangle\}$, x is new	$\exists$
(2)	$\mathcal{L}(x) := \mathcal{L}(x) \cup \{\langle\{o\}, 1\rangle\}$	$\{o\}_2$
(3)	$\mathsf{Merge}(x, r_o)$	$\{o\}_1$

Subsequently, (step (4)) suppose that the $\sqsubseteq$-rule is applied to the axiom $\{o\} \sqsubseteq \leq 3R.\top$ adding either $\langle\neg\{o\}, 1-n+\varepsilon\rangle$ or $\langle\leq 3R.\top, n\rangle$ for all $n \in N^{\mathcal{A}}$ to the labels of all nodes. Consider node r_o and that the algorithm has selected value $n = 0.6$. In that case adding $\langle\neg\{o\}, 0.4+\varepsilon\rangle$ to $\mathcal{L}(r_o)$ creates a clash since $\mathcal{L}(r_o)$ already contains $\langle\{o\}, 1\rangle$. Hence, it proceeds by adding $\langle\leq 3R.\top, 0.6\rangle$ to $\mathcal{L}(r_o)$. Then, (step (5)) the *NN*-rule is applied; this is because $\langle\leq 3R.\top, 0.6\rangle \in \mathcal{L}(r_o)$ and r_o is a nominal node for which $R_c^{\mathbf{G}}(r_o, 0.6, \top) > 1$. (The latter is due to the merging of x and r_o, which made x_a an $R_{0.6}$-neighbour of r_o and $\langle x_a, r_o\rangle$ is conjugated with $\langle\neg R^-, 0.6\rangle$) Thus, the algorithm guesses m (say $m = 3$) and creates three nodes z_i, setting $\mathcal{L}(\langle r_o, z_i\rangle) := \{\langle R, 1-0.6+\varepsilon\rangle\}$, $\mathcal{L}(z_i) := \{\langle\top, 1-0.6+\varepsilon\rangle, \langle\{o_i\}, 1\rangle\}$ for o_i new in $\mathbf{G}$ and $z_i \neq z_j$ for $1 \leq i < j \leq 3$. Then, we have the following application of tableaux rules:

(6)	$\mathsf{Merge}(x_a, z_1)$	$\leq_{\{o\}}$
(7)	$\mathcal{L}(z_1) := \mathcal{L}(z_1) \cup \{\langle\neg C, 0.6\rangle, \langle\exists P^-.(C \sqcap \leq 1P), 0.6\rangle$ $\langle\forall S^-.(\exists P^-.(C \sqcap \leq 1P)), 0.6\rangle\}$	$\sqcap$
(8)	$\mathcal{L}(\langle z_1, y_1\rangle) := \{\langle P^-, 0.6\rangle\}, \mathcal{L}(y_1) := \{\langle C \sqcap \leq 1P, 0.6\rangle\}$	$\exists$
(9)	$\mathcal{L}(y_1) := \mathcal{L}(y_1) \cup \{\langle C, 0.6\rangle, \langle\leq 1P, 0.6\rangle\}$	$\sqcap$
(10)	$\mathcal{L}(y_1) := \mathcal{L}(y_1) \cup \{\langle\exists P^-.(C \sqcap \leq 1P), 0.6\rangle\}$	$\forall$
(11)	$\mathcal{L}(y_1) := \mathcal{L}(y_1) \cup \{\langle\forall S^-.(\exists P^-.(C \sqcap \leq 1P)), 0.6\rangle\}$	$\forall_+$
(12)	$\mathcal{L}(y_1) := \mathcal{L}(y_1) \cup \{\langle\exists R^-.\{o\}, 0.6\rangle\}$	$\sqsubseteq$
(13)	$\mathcal{L}(\langle y_1, y_2\rangle) := \{\langle R^-, 0.6\rangle\}, \mathcal{L}(y_2) \cup \{\langle\{o\}, 0.6\rangle\}$	$\exists$

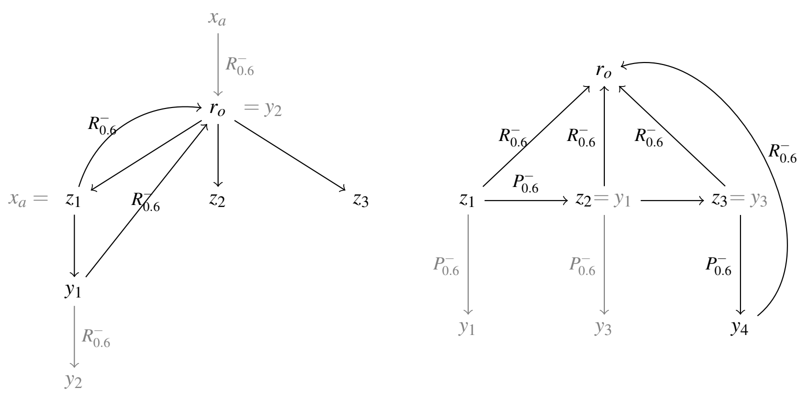

Fig. 1: Illustration of the expansion process

At this point, the $\{o\}_2$ and $\{o\}_1$-rule will be applied to y_2 merging it into r_o, as in steps (2) and (3) for x. Thus, $\mathcal{L}(\langle y_1, r_o\rangle) = \{\langle R^-, 0.6\rangle\}$ is created and again the $\leq$-rule would be applied to r_o merging y_1 into either z_2 or z_3. Note that if it merged y_1 into z_1, then $\mathcal{L}(z_1)$ would contain a clash since $\{\langle \neg C, 0.6\rangle, \langle C, 0.6\rangle\} \subseteq \mathcal{L}(z_1)$ and we would need to backtrack to merge y into one of z_2, z_3.

Hence, assume that y_1 is merged into z_2. Then, z_2 becomes an $P^-_{0.6}$-neighbour of z_1 and $\mathcal{L}(z_2)$ now contains $\{\langle \exists P^-.(C \sqcap \leq 1P), 0.6\rangle, \langle \forall S^-.(\exists P^-.(C \sqcap \leq 1P)), 0.6\rangle\}$. Consequently, similar tableaux rules as in steps (8)–(13) would be applied for z_2—that is, a neighbour y_3 would be created which would eventually be merged into z_3 making z_2 an $P^-_{0.6}$-neighbour of z_3. Similarly as before, if y_3 was merged in either z_1 or z_2 we would have a clash. The latter clash would be because the merge would make z_2 an $P^-_{0.6}$-neighbour of itself, thus it would have two $P_{0.6}$-neighbours which leads to a clash due to $\langle \leq 1P, 0.6\rangle$. Subsequently, following similar steps, a new neighbour y_4 of z_3 would be created, which then leads to a clash since it cannot be merged with any node of the completion-graph. Finally, the algorithm backtracks to the choice of m and selects a smaller value, but obviously this would again lead to a clash following similar reasoning. $\dashv$

Figure 1 depicts the most important parts of the completion-graph of the above example for two different points in time. Note that for clarity reasons we do not show the node and edge labels. The leftmost part depicts the graph until step (11). Originally r_o was an $R^-_{0.6}$-neighbour of x_a, due to the merge of x into r_o (step (3)). Then, the NN-rule applied and created z_1, z_2 and z_3, thus r_o contained four $R_{0.6}$-neighbours causing the $\leq$-rule to be applied merging x_a into z_1, and removing x_a. Then, expansion of node z_1 that contains $\langle F, 0.6\rangle$ creates node y_1 and then also y_2 is created. Then, the algorithm merges y_2 into r_o making y_1 an $R_{0.6}$-neighbour of r_o. Then, again r_o has 4 $R_{0.6}$-neighbours and thus y_1 needs to be merged into z_2.

The rightmost part illustrates the completion-graph from step (14) until the end. Note that we have inverted the edges from r_o to all z_i for clarity reasons and to stress the fact that r_0 has four incoming R^- edges. Similarly to steps (8)–(13) the algorithm would create node y_3 which will be eventually merged into z_3. Finally, also y_4 is created, and for similar reasons it needs to be merged with an existing node, but all choices lead to a clash.

Correctness and termination of the tableaux-based algorithm for deciding satisfiability of f_{KD}-$\mathcal{SHOIQ}$ knowledge bases is made formal in the lext lemma which is proven in the appendix.

Lemma 2 *Let Σ be an f_{KD}-$\mathcal{SHOIQ}$ knowledge base. Then*

1. ***Termination**: when started for Σ the tableaux algorithm terminates.*
2. ***Correctness**: Σ has a fuzzy tableau iff the tableaux algorithm for f_{KD}-$\mathcal{SHOIQ}$ can be applied to Σ such that it yields a complete and clash-free completion-graph.*

5 A Reasoning Algorithm for fuzzy nominals

Although the fuzzy nominal constructor was initially proposed in [5] the authors did not present a tableaux decision procedure for handling them. Instead they presented a reduction technique that translates an f-$\mathcal{SHO}_{\mathtt{f}}\mathcal{IN}$ knowledge base into a (crisp) $\mathcal{SHOIN}$ knowledge base. Preliminary evaluation [16] has shown that the additional axioms needed in order to reduce the fuzzy knowledge base affect the reasoning performance. In constrast, direct tableaux algorithms can be directly optimised and exhibit acceptable performance [44]. In the following we will modify the reasoning algorithm of f_{KD}-$\mathcal{SHOIQ}$ in order to provide a direct tableaux-based reasoning algorithm that decides f_{KD}-$\mathcal{SHO}_{\mathtt{f}}\mathcal{IQ}$ knowledge base satisfiability.

First, we need to modify Definition 2 and more precisely Properties 14 and 15 since an element $s \in \mathbf{S}$ of the fuzzy tableau can belong to a fuzzy nominal to any degree from $(0,1]$.

Definition 5 Let Σ be an f_{KD}-$\mathcal{SHO}_{\mathtt{f}}\mathcal{IQ}$ knowledge base and X^{Σ} as defined in Section 3.2. A *fuzzy tableau* T for Σ is defined similarly to an f_{KD}-$\mathcal{SHOIQ}$ fuzzy tableau for Σ but instead of Properties 14–16 of Definition 3.2 it satisfies the following (modified) properties:

14'. If $\mathcal{L}(x,\{o,n\}) = n$ and $\mathcal{L}(y,\{o,n\}) = n$, for some $o \in \mathbf{I}$ and $n \in (0,1]$, then $x = y$.

15'. If $\mathcal{L}(x,\{o,n\}) \rhd k$, with $k \rhd^{-} n$, then $\mathcal{L}(x,\{o,n\}) = n$.

16'. If $C \sqsubseteq D \in \mathcal{T}$, then either $\mathcal{L}(s,\neg C) > 1-n$ or $\mathcal{L}(s,D) \geq n$, for all $s \in \mathbf{S}$ and $n \in X^{\Sigma}$

$\diamondsuit$

Lemma 3 *An f_{KD}-$\mathcal{SHO}_{\mathtt{f}}\mathcal{IQ}$ knowledge base Σ is satisfiable iff there exists a fuzzy tableau for Σ.*

Proof The modified Properties 14' and 15' ensure the correct interpretation of fuzzy nominals, while Property 16' ensures that the model induced by T is a model of the TBox $\mathcal{T}$. Hence, the claim follows straightforwardly by the proof of Lemma 1. $\square$

As a consequence, some modification in the definitions of completion-graph and tableaux algorithm for f_{KD}-$\mathcal{SHOIQ}$ knowledge bases are required in order to support fuzzy nominals. More precisely, we need to modify the tableaux expansion rules, the clash conditions, and the initialisation of the completion-graph.

Definition 6 A completion-graph $\mathbf{G}$ for an f_{KD}-$\mathcal{SHO}_{\mathtt{f}}\mathcal{IQ}$ knowledge base Σ is similar to an f_{KD}-$\mathcal{SHOIQ}$ completion-graph but with the following modifications:

For some node x, we say that $\mathcal{L}(x)$ contains a *clash* if it satisfies one of the clash conditions of f_{KD}-$\mathcal{SHOIQ}$ in Definition 3, or if one of the following holds:

Table 4: Tableaux rules for the fuzzy nominals of f$_{KD}$-$\mathcal{SHO}_{\mathtt{f}}\mathcal{IQ}$

Rule		Description
$\{o,n\}_1$	if	for some $\{o,n\}$ there exist two nodes x,y with $\langle\{o,n\},k\rangle \in \mathcal{L}(x)\cap\mathcal{L}(y)$ and not $x \neq y$
	then	$\mathsf{Merge}(x,y)$
$\{o,n\}_2$	if	$\langle\{o,n\},k\rangle \in \mathcal{L}(x)$, and $\{\langle\{o,n\},n\rangle,\langle\neg\{o,n\},1-n\rangle\} \not\subseteq \mathcal{L}(x) = \emptyset$
	then	$\mathcal{L}(x) := \mathcal{L}(x) \cup \{\langle\{o,n\},n\rangle,\langle\neg\{o,n\},1-n\rangle\}$

- for some $o \in \mathbf{I}$ and $n \in (0,1]$, there exist nodes $x \neq y$ with $\langle\{o,n\},n\rangle \in \mathcal{L}(x)\cap\mathcal{L}(y)$.
- $\mathcal{L}(x)$ contains a pair of the form $\langle\{o,n\},k\rangle$ with $k > n$.

A completion-graph for f$_{KD}$-$\mathcal{SHO}_{\mathtt{f}}\mathcal{IQ}$ is initialised in a similar way as a completion-graph for f$_{KD}$-$\mathcal{SHOIQ}$ but each $\langle\{a_i\},1\rangle$ in the initialisation is replaced to the equivalent fuzzy nominal $\langle\{a_i,1\},1\rangle$ and additionally, for each fuzzy nominal $\{o_i,n\}$ that appears in Σ a nominal node r_{o_i} with $\mathcal{L}(r_i) = \{\langle\{o_i,n\},n\rangle,\langle\neg\{o_i,n\},1-n\rangle\langle\{o_i,1\},1\rangle\}$ is created. Finally, Table 4 presents the tableaux expansion rules that are needed for correctly handling fuzzy nominals. ♢

It is interesting to note that, according to the above, if for some $o \in \mathbf{I}$ and node x we have $\{\langle\{o,0.5\},0.5\rangle,\langle\{o,0.7\},0.7\rangle\} \subseteq \mathcal{L}(x)$, then $\mathcal{L}(x)$ does not contain a clash. This is in accordance to the semantics as $\{o,0.5\}$ denotes a different fuzzy concept than $\{o,0.7\}$ and hence for $x = o$ the constraints are satisfied. In general any node can belong to an infinite number of fuzzy nominal concepts of the form $(\neg)\{o,n_i\}$ for $n_i \in (0,1]$, since different degrees imply a different fuzzy set.

The following follows straightforwardly from the proof of correctness of the algorithm for f$_{KD}$-$\mathcal{SHOIQ}$ and the modified Definition 3.

Lemma 4 *Let Σ be an f$_{KD}$-$\mathcal{SHO}_{\mathtt{f}}\mathcal{IQ}$ knowledge base. Then*

1. ***Termination**: when started for Σ the tableaux algorithm terminates.*
2. ***Correctness**: Σ has a fuzzy tableau iff the tableaux algorithm for f$_{KD}$-$\mathcal{SHO}_{\mathtt{f}}\mathcal{IQ}$ can be applied to Σ such that it yields a complete and clash-free completion-graph.*

In the following we present some examples of the reasoning algorithm for fuzzy nominals. First, we show how the new rules work with a simple example.

Example 3 Let the knowledge base $\Sigma = \langle\mathcal{T},\mathcal{A}\rangle$, where $\mathcal{T}$ and $\mathcal{A}$ are defined as follows:

$$\mathcal{T} = \{\mathsf{GermanSpeaking} \equiv \{germany,1\} \sqcup \{austria,1\} \sqcup \{switzerland,0.67\}\}$$
$$\mathcal{A} = \{(a : \mathsf{GermanSpeaking}) = 0.67\}$$

We want to prove that a is Switzerland—that is, whether $\Sigma \models a \doteq switzerland$, which is equivalent to checking whether $\Sigma \models a : \{switzerland,1\} \geq 1$. This query is reduced to checking the unsatisfiability of $\Sigma' = \langle\mathcal{T},\mathcal{A}\cup\{a : \neg\{switzerland,1\} \geq \varepsilon\}\rangle$, for some value $\varepsilon > 0$.

Firstly, the algorithm initialises a completion-graph $\mathbf{G}$ to contain the following nodes with the respective node labels:

$$(1) \quad \mathcal{L}(x_a) := \{\langle\mathsf{GermanSpeaking},=,0.67\rangle,\langle\{a,1\},1\rangle,\langle\neg\{switz,1\},\varepsilon\rangle\}$$
$$(2) \quad \mathcal{L}(r_{germ}) := \{\langle\{germ,1\},1\rangle,\langle\neg\{germ,1\},0\rangle\}$$
$$(3) \quad \mathcal{L}(r_{aus}) := \{\langle\{aus,1\},1\rangle,\langle\neg\{aus,1\},0\rangle\}$$
$$(4) \quad \mathcal{L}(r_{switz}) := \{\langle\{switz,0.67\},0.67\rangle,\langle\neg\{switz,0.67\},0.33\rangle,\langle\{switz,1\},1\rangle\}$$

Table 5: Expansion of completion-graph from Example 3

(5)	$\mathcal{L}(x_a) := \mathcal{L}(x_a) \cup \{\langle\neg\{ger,1\},0.33\rangle,\langle\neg\{aus,1\},0.33\rangle,$ $\langle\neg\{switz,0.67\},0.33\rangle\}$	$2\times\sqcap : (1)$
$(6a)$	$\mathcal{L}(x_a) := \mathcal{L}(x_a) \cup \{\langle\{ger,1\},0.67\rangle\}$	$\sqcup : (1)$
$(6b)$	$\mathcal{L}(x_a) := \mathcal{L}(x_a) \cup \{\langle\{aus,1\},0.67\rangle\}$	$\sqcup : (1)$
$(6c)$	$\mathcal{L}(x_a) := \mathcal{L}(x_a) \cup \{\langle\{switz,0.67\},0.67\rangle\}$	$\sqcup : (1)$
$(7a)$	$\mathcal{L}(x_a) := \mathcal{L}(x_a) \cup \{\langle\{ger,1\},1\rangle,\langle\neg\{ger,1\},0\rangle\}$	$\{o,n\}_2 : (6a)$
$(7b)$	$\mathcal{L}(x_a) := \mathcal{L}(x_a) \cup \{\langle\{aus,1\},1\rangle,\langle\neg\{aus,1\},0\rangle\}$	$\{o,n\}_2 : (6b)$
$(7c)$	$\mathcal{L}(x_a) := \mathcal{L}(x_a) \cup \{\langle\{switz,0.67\},0.67\rangle,\langle\neg\{switz,0.67\},0.33\rangle\}$	$\{o,n\}_2 : (6c)$
$(8a)$	$\mathcal{L}(x_a)$ contains a clash: $\langle\{ger,1\},1\rangle$ and $\langle\neg\{ger,1\},0.33\rangle$	
$(8b)$	$\mathcal{L}(x_a)$ contains a clash: $\langle\{aus,1\},1\rangle$ and $\langle\neg\{aus,1\},0.33\rangle$	
$(8c)$	$\mathsf{Merge}(x_a, r_{switz})$	$\{o,n\}_1 : (7c),(3)$
$(9b)$	$\mathcal{L}(x_a)$ contains a clash: $\langle\{switz,1\},1\rangle$ and $\langle\neg\{switz,1\},\varepsilon\rangle$	

and also the relation $\neq$ to be empty.

Due to the equivalence axiom of the TBox, concept GermanSpeaking in the label of x_a is replaced with its definition. Moreover, $(a : \mathsf{GermanSpeaking}) = 0.67$ is equivalent to two fuzzy assertions of the form $(a : \mathsf{GermanSpeaking}) \geq 0.67$ and $(a : \mathsf{GermanSpeaking}) \leq 0.67 \to (a : \neg\mathsf{GermanSpeaking}) \geq 0.33$. Hence, we finally have the following:

$$\langle\{ger,1\}\sqcup\{aus,1\}\sqcup\{switz,0.67\},0.67\rangle \in \mathcal{L}(x_a) \text{ and}$$
$$\langle\neg\{ger,1\}\sqcap\neg\{aus,1\}\sqcap\neg\{switz,0.67\},0.33\rangle \in \mathcal{L}(x_a)$$

Then, we repeatedly apply the rules in Tables 3 and 4, obtaining the expansion depicted in Table 5, where the application of rule $\sqcup$ creates 3 different alternatives, namely $6a$, $6b$ and $6c$. ⊣

The above example illustrates the importance of the $\{o,n\}_2$-rule for correctly identifying the inconsistencies that exist within the completion-graph.

Next we give an example that shows reasoning over general TBox axioms and fuzzy nominals.

Example 4 Let the following f$_{KD}$-$\mathcal{SHO}_{\mathfrak{f}}\mathcal{IQ}$ TBox:

$$\mathcal{T} = \{\{greece,1\}\sqcup\{albania,0.8\}\sqcup\{montenegro,0.7\}\sqcup\{croatia,0.6\}\sqsubseteq \mathsf{VulcanMed}\}.$$

Obviously, $\mathcal{T}$ entails the fuzzy assertion $(albania : \mathsf{VulcanMed}) \geq 0.8$. We can now use the rules for fuzzy nominals together with the technique for handling GCIs in order to answer this query. Firsly, we reduce it to the problem of checking the unsatisfiability of the ABox $\mathcal{A} = \{(albania : \neg\mathsf{VulcanMed}) \geq 0.2+\varepsilon\}$. Then we construct the set of relevant membership degrees which is the set. $N^{\Sigma} = \{0,0.2,0.2+\varepsilon,0.3,0.4,0.5,0.6,0.7,0.8-\varepsilon,0.8,1\}$ and consequently we initialise a completion-graph to contain the following nodes with the respective node labels (for brevity reasons we do not show all of the nominal nodes; these are initialised in a similar way):

(1) $\mathcal{L}(x_{alb}) := \{\langle\neg\mathsf{VulcanMed},0.2+\varepsilon\rangle,\langle\{alb,1\},1\rangle\}$

(2) $\mathcal{L}(r_{alb}) := \{\langle\{alb,0.8\},0.8\rangle,\langle\neg\{alb,0.8\},0.2\rangle,\langle\{alb,1\},1\rangle\}$

Subsequently, we repeatedly apply the expansion rules of Tables 3 and 4. Then, we obtain the following steps of rule applications:

$$
\begin{array}{lll}
(3) & \mathsf{Merge}(x_{alb}, r_{albania}) & \{o,n\}_1 : (1),(2) \\
(4a) & \mathcal{L}(x_{alb}) := \mathcal{L}(x_{alb}) \cup \{\langle \neg\{gre,1\}, 1-n+\varepsilon\rangle, & \\
 & \quad \langle \neg\{alb,0.8\}, 1-n+\varepsilon\rangle, & \sqsubseteq: (3) \\
 & \quad \langle \neg\{mont,0.7\}, 1-n+\varepsilon\rangle, & \\
 & \quad \langle \neg\{cro,0.6\}, 1-n+\varepsilon\rangle\} & \\
(4b) & \mathcal{L}(x_{alb}) := \mathcal{L}(x_{alb}) \cup \{\langle \mathsf{VulcanMed}, n\rangle\} &
\end{array}
$$

Similar disjunction as above (steps 4a and 4b) will be performed for every $n \in N^{\Sigma}$. Consequently, at some point we will chose the value $n = 0.8$ and both alternatives will lead to a clash. At step $(4b)$ node $\mathcal{L}(x_{alb})$ would contain a clash due to the pairs $\{\langle \neg\mathsf{VulcanMed}, 0.2+\varepsilon\rangle, \langle \mathsf{VulcanMed}, 0.8\rangle\} \subseteq \mathcal{L}(x_{alb})$, while at step $(4a)$ $\mathcal{L}(x_{alb})$ also contains a clash due to the pairs $\{\langle \neg\{alb,0.8\}, 1-n+\varepsilon\rangle, \langle \{alb,0.8), 0.8\rangle\} \subseteq \mathcal{L}(x_{alb})$ since $n = 0.8 \Rightarrow 1-n+\varepsilon = 0.2+\varepsilon > 0.2$, consequently $1-n+0.8 > 1$. ⊣

6 Extending Reasoning to f$_{KD}$-$\mathcal{SROIQ}$

In this section we extend the tableaux reasoning algorithm we presented in the previous sections further, providing reasoning support for f$_{KD}$-$\mathcal{SROIQ}$ knowledge bases. $\mathcal{SROIQ}$ is a very expressive DL [31] forming the logical underpinnings of the relatively new W3C standard OWL 2 DL [38]. In the following we first recapitulate the syntax and semantics of f$_{KD}$-$\mathcal{SROIQ}$ [7] and then present our reasoning approach.

Consider again an alphabet of distinct concept names (**C**), role names (**R**) together with the universal role U [31],[6] and individuals (**I**). As with f$_{KD}$-$\mathcal{SHOIQ}$, f$_{KD}$-$\mathcal{SROIQ}$-roles are defined (inductively) by the syntax $S \to RN \mid R^-$, where $RN \in \mathbf{R}$ and R^- represents the *inverse* of R. Let $A \in \mathbf{C}$, $R \in \mathbf{R}$, $o \in \mathbf{I}$, and $p \in \mathbb{N}$; then, *f$_{KD}$-$\mathcal{SROIQ}$-concepts* are defined inductively as follows:

$$C,D \longrightarrow \bot \mid \top \mid A \mid C \sqcup D \mid C \sqcap D \mid \neg C \mid \forall R.C \mid \exists R.C \mid \geq pR.C \mid \leq pR.C \mid \{o\} \mid \exists R.\mathsf{Self}$$

As can be noted, f$_{KD}$-$\mathcal{SROIQ}$-concepts are defined similarly as f$_{KD}$-$\mathcal{SHOIQ}$-concepts with the additional concept constructor $\exists R.\mathsf{Self}$. Moreover, note again that in concepts of the form $\geq pR.C$, $\leq pR.C$, and $\exists R.\mathsf{Self}$, R needs to be simple. In the case of f$_{KD}$-$\mathcal{SROIQ}$ the definition of simple roles is more involved and is given below.

Let $R_1,\ldots,R_n,S$ with $n \geq 1$ be f$_{KD}$-$\mathcal{SROIQ}$-roles. A *role hierarchy* $\mathcal{R}_h$ is a set of *complex role inclusion axioms* (cRIAs) of the form $R_1 \ldots R_n \sqsubseteq S$. Intuitively, such axioms state that the composition of roles $R_1,\ldots,R_n$ imply the role S. We often use the notation $w \sqsubseteq R$, where w is a finite string of roles not including U. For $R,S \neq U$ f$_{KD}$-$\mathcal{SROIQ}$-roles, $\mathcal{R}_a$ is a set of *role properties* of one of the following forms: $\mathsf{Trans}(R)$, $\mathsf{Ref}(R)$, $\mathsf{Irr}(R)$, $\mathsf{Sym}(R)$, $\mathsf{ASym}(R)$, and $\mathsf{Dis}(R,S)$. Intuitively, these axioms state that R is transitive, reflexive, irreflexive, symmetric, antisymmetric, and disjoint from S, respectively. Next, we define the notion of simple roles in f$_{KD}$-$\mathcal{SROIQ}$.

Definition 7 ([31]) Given a role hierarchy $\mathcal{R}_h$ and a set of role properties $\mathcal{R}_a$ the set of *simple* roles is inductively defined as follows:

[6] Intuitively, the universal role is a role that connects all objects of $\Delta^{\mathcal{I}}$ with each other.

- A role name R in $\mathcal{R}_h$ or $\mathcal{R}_a$ is simple if it does not occur on the right hand side of a cRIA in $\mathcal{R}_h$.
- A role R^- is simple if R is simple, and
- If R occurs on the right-hand aide of a cRIA in $\mathcal{R}_h$, then R is simple if, for each $w \sqsubseteq R \in \mathcal{R}_h$, $w = S$ and S is simple.

Apart from undecidability stemming from the use of non-simple roles in concepts of the form $\geq pR.C$, $\leq pR.C$, and $\exists R.\mathsf{Self}$, cRIAs impose new (un)decidability issues. To ensure decidability the role hierarchy needs to be *regular* [31] in the sense defined below.

Definition 8 ([31]) A strict partial order $\prec$ over a set $\mathcal{V}$ is an irreflexive and transitive relation on $\mathcal{V}$. A strict partial order $\prec$ over the set of f_{KD}-$\mathcal{SROIQ}$ roles $\mathbf{R} \cup \{R^- \mid R \in \mathbf{R}\}$ is called *regular* if $S \prec R \Longleftrightarrow S^- \prec R$.

Let $\prec$ be a regular order over f_{KD}-$\mathcal{SROIQ}$-roles. A cRIA $w \sqsubseteq R$ is $\prec$-regular if $R \in \mathbf{R}$, and

1. $w = RR$, or
2. $w = R^-$, or
3. $w = S_1 \dots S_n$ and $S_i \prec R$ for all $1 \leq i \leq n$, or
4. $w = RS_1 \dots S_n$ and $S_i \prec R$ for all $1 \leq i \leq n$, or
5. $w = S_1 \dots S_n R$ and $S_i \prec R$ for all $1 \leq i \leq n$.

Finally, a role hierarchy $\mathcal{R}_h$ is called *regular* if there exists a regular order $\prec$ such that each cRIA in $\mathcal{R}_h$ is $\prec$-regular. $\diamondsuit$

Given the above, an f_{KD}-$\mathcal{SROIQ}$ *RBox* consists of a pair $\langle \mathcal{R}_h, \mathcal{R}_a \rangle$, where $\mathcal{R}_h$ is a regular role hierarchy and $\mathcal{R}_a$ is a finite set of role properties such that the roles that appear in axioms of the form $\mathsf{Irr}(R)$, $\mathsf{ASym}(R)$, and $\mathsf{Dis}(R,S)$ are simple.

An f_{KD}-$\mathcal{SROIQ}$ *TBox* is defined exactly as in f_{KD}-$\mathcal{SHOIQ}$. An f_{KD}-$\mathcal{SROIQ}$ *ABox* is a set of fuzzy assertions and (in)equality axioms (like in f_{KD}-$\mathcal{SHOIQ}$), as well as fuzzy assertions of the form $\langle (a,b) : \neg R \rangle \geq n$ called *simple* role negations [31]. Note, however, that the latter are simply syntactic sugar of f_{KD}-$\mathcal{SHOIQ}$ assertions of the form $\langle (a,b) : R \rangle \leq 1 - n$ which, for the purposes of reasoning, can then be normalised as shown in Section 3.1. In addition, in all concepts of the form $\exists R.\mathsf{Self}$, $\geq pR.C$, and $\leq pR.C$ that appear in either $\mathcal{T}$ or $\mathcal{A}$, R is a simple role.

Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a fuzzy interpretation as defined in Section 3. The semantics of f_{KD}-$\mathcal{SROIQ}$-concepts are given by the equations depicted in Table 2 together with the equation for concept $\exists R.\mathsf{Self}$ depicted in the upper part of Table 6. Since f_{KD}-$\mathcal{SROIQ}$ TBoxes and ABoxes are exactly as in f_{KD}-$\mathcal{SHOIQ}$ the definition of models is exactly the same. However, a fuzzy interpretation $\mathcal{I}$ satisfies an f_{KD}-$\mathcal{SROIQ}$ RBox if $\mathcal{I}$ satisfies each axiom in $\mathcal{R}_h$ and each axiom in $\mathcal{R}_a$ as shown in the lower part of Table 6. More precisely, $\mathcal{I}$ satisfies an axiom of the left column of the lower part of Table 6 if the condition in the right column holds, where $R, R_i, S \in \mathbf{R}$ for $1 \leq i \leq m$ and $a, b \in \Delta^{\mathcal{I}}$; satisfaction of axioms of the form $\mathsf{Trans}(R)$ are as described for f_{KD}-$\mathcal{SHOIQ}$ in Section 3.1.

As we can see cRIAs are interpreted as the sup-t composition of fuzzy relations, which due to its associativity property [33] can be applied to any number of relations. Moreover, from the properties of the sup-t composition [33] and the semantics of inverse roles it holds that if $\mathcal{I}$ satisfies $R_1 \dots R_n \sqsubseteq S$, then it also satisfies $\mathsf{Inv}(R_n) \dots \mathsf{Inv}(R_1) \sqsubseteq \mathsf{Inv}(S)$.

Next, we show that similarly to crisp $\mathcal{SROIQ}$ [31], several role properties can be encoded with the aid of cRIAs or by using the new special concept constructor $\exists R.\mathsf{Self}$.

Table 6: Semantics of Additional constructors and role axioms of f$_{KD}$-$\mathcal{SROIQ}$

Concept	**Semantics**
$\exists R.\mathsf{Self}$	$(\exists R.\mathsf{Self})^{\mathcal{I}}(a) = R^{\mathcal{I}}(a,a)$
Role Axiom	**Semantics**
$\mathsf{Ref}(R)$	$R^{\mathcal{I}}(a,a) = 1$
$\mathsf{Irr}(R)$	$R^{\mathcal{I}}(a,a) = 0$
$\mathsf{Sym}(R)$	$R^{\mathcal{I}}(a,b) = R^{\mathcal{I}}(b,a)$
$\mathsf{ASym}(R)$	$\min(R^{\mathcal{I}}(a,b), R^{\mathcal{I}}(b,a)) = 0$
$\mathsf{Dis}(R,S)$	$\min(R^{\mathcal{I}}(a,b), S^{\mathcal{I}}(a,b)) = 0$
$R_1 \dots R_m \sqsubseteq S$	$[R_1^{\mathcal{I}} \circ^t \dots \circ^t R_m^{\mathcal{I}}](a,b) \leq S^{\mathcal{I}}(a,b)$

Proposition 1 (Reduction of Role Properties) *Without loss of generality we can assume that f$_{KD}$-$\mathcal{SROIQ}$ RBoxes do not contain role properties of the form* $\mathsf{Irr}(R)$*,* $\mathsf{Ref}(R)$*,* $\mathsf{Sym}(R)$*, or* $\mathsf{Trans}(R)$.

Proof For the case of role properties of the form $\mathsf{Sym}(R)$ it has been shown already that any fuzzy interpretation $\mathcal{I}$ satisfies $\mathsf{Sym}(R)$ iff it satisfies both $R^- \sqsubseteq R$ and $R \sqsubseteq R^-$ [47]. Now let $\mathcal{I}$ be a fuzzy interpretation. Then,

- If $\mathcal{I}$ satisfies $\mathsf{Trans}(R)$, then for all $a,b \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a,c) \geq \sup_{b\in\Delta^{\mathcal{I}}}\{t(R^{\mathcal{I}}(a,b), R^{\mathcal{I}}(b,c))\}$. Clearly, this consists of a sup-t composition of the role R with itself, hence $\mathcal{I}$ satisfies $\mathsf{Trans}(R)$ iff it satisfies $[R^{\mathcal{I}} \circ^t R^{\mathcal{I}}](a,b) \sqsubseteq R^{\mathcal{I}}(a,b)$ and, clearly, we can replace $\mathsf{Trans}(R)$ with the role inclusion axiom $RR \sqsubseteq R$.
- If $\mathcal{I}$ satisfies $\mathsf{Ref}(R)$, then for all $a \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a,a) = 1$. Let a be an arbitrary object of $\Delta^{\mathcal{I}}$. Subsequently, we have the following equivalences: $R^{\mathcal{I}}(a,a) = 1 \Leftrightarrow R^{\mathcal{I}}(a,a) \geq 1 \Leftrightarrow (\exists R.\mathsf{Self})^{\mathcal{I}}(a) \geq \top^{\mathcal{I}}(a)$. Since a is arbitrary, the last inequality holds for all objects in $\Delta^{\mathcal{I}}$. Hence, we can abstract from interpretations and replace $\mathsf{Ref}(R)$ with $\top \sqsubseteq \exists R.\mathsf{Self}$.
- If $\mathcal{I}$ satisfies $\mathsf{Irr}(R)$, then for all $a \in \Delta^{\mathcal{I}}.R^{\mathcal{I}}(a,a) = 0$. Similarly, for some $a \in \Delta^{\mathcal{I}}$ we have the following: $R^{\mathcal{I}}(a,a) = 0 \Leftrightarrow R^{\mathcal{I}}(a,a) \leq 0 \Leftrightarrow c((\exists R.\mathsf{Self})^{\mathcal{I}})(a) \geq c(0) \Leftrightarrow (\neg\exists R.\mathsf{Self})^{\mathcal{I}}(a) \geq 1 \Leftrightarrow (\neg\exists R.\mathsf{Self})^{\mathcal{I}}(a) \geq \top^{\mathcal{I}}(a)$. Thus, we can replace $\mathsf{Irr}(R)$ with $\top \sqsubseteq \neg\exists R.\mathsf{Self}$. □

Finally, as shown in [31], for the purposes of deciding satisfiability in $\mathcal{SROIQ}$ the universal role U can be simulated by a *pseudo-universal* role U'. More precisely, given an f$_{KD}$-$\mathcal{SROIQ}$ KB Σ we can replace all occurrences of U in Σ with U', then add the role properties $\{\mathsf{Trans}(U'), \mathsf{Sym}(U'), \mathsf{Ref}(U')\}$ to $\mathcal{R}_a$ and, finally, add a role inclusion axiom of the form $R \sqsubseteq U'$ to $\mathcal{R}_h$ for each R that occurs in Σ.

6.1 Reasoning in f$_{KD}$-$\mathcal{SROIQ}$

A major issue in providing reasoning support for $\mathcal{SROIQ}$ is the management of complex role inclusion axioms. More precisely, if we have the assertion $a : \forall R.C \in \mathcal{A}$ together with cRIAs of the form $w \sqsubseteq R$, i.e., that have R in the right-hand side, then the algorithm must ensure that C is properly 'propagated' along paths of roles that possibly appear in the ABox and which imply the existence of role R. For example, from the assertions $a : \forall R.C, (a,b) :$

$R_1,(b,c):R_2$, and axioms $R_1R_2 \sqsubseteq S, S \sqsubseteq R$ one should infer $c:C$, i.e., propagate C along the path $(a,b):R_1,(b,c):R_2$. This is because, for any interpretation $\mathcal{I}$ satisfying the above axioms we must have $\langle a^{\mathcal{I}},c^{\mathcal{I}}\rangle \in S^{\mathcal{I}}$ and $\langle a^{\mathcal{I}},c^{\mathcal{I}}\rangle \in R^{\mathcal{I}}$; hence, since $a^{\mathcal{I}} \in (\forall R.C)^{\mathcal{I}}$ we must have $c^{\mathcal{I}} \in C^{\mathcal{I}}$. In the presence of many cRIAs it could be quite complicated to keep track of which concept needs to be propagated along which path. To provide with a systematic way to encode *all* possible paths that can imply the existence of a role, the use of non-deterministic finite automata has been proposed in the literature [18,32]. Intuitively, an automaton $\mathcal{B}_R$ for a role R memorises the paths that are implied by the axioms in $\mathcal{R}_h$ and have R in the righ-hand side.

Example 5 Consider the following role hierarchy:

$$\mathcal{R}_h = \{R_1R_2 \sqsubseteq R, R \sqsubseteq S, SS \sqsubseteq S\}$$

According to the automata construction method presented in [32], the automaton $\mathcal{B}_R$ for R w.r.t. $\mathcal{R}_h$ would consist of the following transitions:

$$\delta_R = \{i_R \rightarrow_{R_1} s_1,\ s_1 \rightarrow_{R_2} f_R,\ i_R \rightarrow_R f_R\}$$

where i_R, s_1, f_R are the states of $\mathcal{B}_R$, i_R is its initial state, and f_R is its final state. Intuitively, the first two states are due to axiom $R_1R_2 \sqsubseteq R$ and encode the fact that from the initial state of $\mathcal{B}_R$ one can go to its final by a path of the form $(a,b):R_1,(b,c):R_2$.

Finally, the automaton $\mathcal{B}_S$ for S w.r.t. $\mathcal{R}_h$ would consist of the following transitions:

$$\delta_S = \{i_S \rightarrow_R f_S,\ i_S \rightarrow_S f_S,\ f_S \rightarrow i_S,\ i_S \rightarrow i_R,\ f_R \rightarrow f_S\} \cup \delta_R$$

where i_S is its initial and f_S its final states. Again, the first transition is due to axiom $R \sqsubseteq S$. Moreover, due to this axiom $\mathcal{B}_S$ also encodes the fact that we can go from i_S to f_S 'through' the automaton $\mathcal{B}_R$; hence, $\mathcal{B}_S$ includes δ_R and empty transitions $i_S \rightarrow i_R$ and $f_R \rightarrow f_S$. Finally, note that in $\mathcal{B}_S$ we can also go from f_S to i_S using an empty transition ($i_S \rightarrow f_S$). This is due to the axiom $SS \sqsubseteq S$ (recall that this means that S is transitive). ⊣

For the details of the automata construction technique we refer the reader to [32] and don't repeat the construction here. Instead we show the next proposition (see appendix for proof) which is an extension of Proposition 9 from [32] and states that indeed $\mathcal{B}_S$ constructed exactly as in [32] captures all implications between paths of roles that imply S.

Proposition 2 *Let $\mathcal{R}_h$ be a regular role hierarchy, let $\mathcal{B}_S$ be the automaton constructed for a role S w.r.t. $\mathcal{R}_h$ according to the method in [32], and let $L(\mathcal{B}_S)$ denote the language accepted by $\mathcal{B}_S$. Then, $\mathcal{I}$ is a model of $\mathcal{R}_h$ if and only if for each role S occuring in $\mathcal{R}_h$, each word $w \in L(\mathcal{B}_S)$ and each $w^{\mathcal{I}}(a,b) \geq n$ we have $S^{\mathcal{I}}(a,b) \geq n$.*

Given that $\mathcal{B}_S$ correctly captures the semantic restrictions imposed by cRIAs, one can subsequently use $\mathcal{B}_S$ to provide reasoning support for such role axioms. There are two alternative ways to accomplish this. Horrocks and Sattler [32] proposed a direct tableaux-based method which incorporates the automata within the tableaux calculus. During tableaux expansion the algorithm 'reads' the current state and transitions of an automaton and matches it with role assertions in the completion-graph. If certain conditions are met then the automaton changes state and is propagated to subsequent nodes of the completion-graph. Finally, if a final state is reached then a concept C is added to the respective node of the completion-graph, hence simulating the propagation of C. For example, if for an individual a we have $a:\forall S.C$ the algorithm first associates the pair $\langle \mathcal{B}_S, C\rangle$ with individual a and sets $\mathcal{B}_S$ to its initial state. Then, if $(a,b) : R_1$ and there exists a transition $i_S \to_{R_1} s_1 \in \mathcal{B}_S$ (i.e., from the initial state of $\mathcal{B}_S$ with symbol R_1 go to state s_1) the algorithm associates to b the pair $\langle \mathcal{B}_S, C\rangle$ but with $\mathcal{B}_S$ set to state s_1. If $\langle \mathcal{B}_S, C\rangle$ is associated with an individual c and $\mathcal{B}_S$ is in its final state, then $c : C$ is asserted. Such an approach has been implemented in the DL reasoner FaCT++.

Alternatively, it has been shown in [18] that the propagation of $\langle \mathcal{B}_S, C\rangle$ and the change of states can also be simulated by proper (TBox) axioms. In that way the initial knowledge base can be transformed into an equisatisfiable one where $\mathcal{R}_h = \emptyset$. We call f$_{KD}$-$\mathcal{SHOIQ}^+$ knowledge base an f$_{KD}$-$\mathcal{SROIQ}$ knowledge base with an empty role hierarchy.

Definition 9 Let $\Sigma = \langle \mathcal{T}, \langle \mathcal{R}_h, \mathcal{R}_a\rangle, \mathcal{A}\rangle$ be an f$_{KD}$-$\mathcal{SROIQ}$ knowledge base. Then, $\tau(\Sigma)$ is the f$_{KD}$-$\mathcal{SHOIQ}^+$ knowledge base $\langle \mathcal{T}', \langle \emptyset, \mathcal{R}_a\rangle, \mathcal{A}'\rangle$, where $\mathcal{T}'$ and $\mathcal{A}'$ are constructed as follows: For each occurrence of a concept $\forall S.C \in \Sigma$ such that $\mathcal{B}_S$ is non-empty associate a unique index j for this occurrence, replace $\forall S.C$ with the fresh concept i_S^j, where i_S is the initial state of $\mathcal{B}_S$ and do the following:

- For each transition $s_1 \to_R s_2 \in \mathcal{B}_S$, $\mathcal{T}'$ contains the axiom $s_1^j \sqsubseteq \forall R.s_2^j$, where s_1^j, s_2^j are (fresh) concept names.
- For each (empty) transition $s_1 \to s_2 \in \mathcal{B}_S$, $\mathcal{T}'$ contains the axiom $s_1^j \sqsubseteq s_2^j$, where s_1^j, s_2^j are (fresh) concept names.
- For each final state f_S of $\mathcal{B}_S$, $\mathcal{T}'$ contains the axiom $f_S^j \sqsubseteq C$.

♢

Intuitively, if during reasoning an individual a is such that $a : \forall S.C$, then the algorithm would mark a with a concept that denotes the initial state of $\mathcal{B}_S$. Then, axioms of the form $s_1^j \sqsubseteq \forall R.s_2^j$ play the role of state propagation along paths of individuals starting from a. If an individual c is marked with a final state of $\mathcal{B}_S$, then $a : C$ would also be asserted by the reasoning algorithm. Such an approach has been implemented in the DL reasoner HermiT and despite the increase of TBox axioms due to the translation, experimental evaluation has shown that this approach also performs well in practice [22].

Due to Proposition 2 there is strong evidence that either of the previous approaches can be used to provide reasoning support for f$_{KD}$-$\mathcal{SROIQ}$. In the current paper we choose to follow the latter approach and show that the translation given in Definition 9 indeed produces an equisatisfiable f$_{KD}$-$\mathcal{SHOIQ}^+$ knowledge base. Hence, to provide reasoning support for f$_{KD}$-$\mathcal{SROIQ}$ one can simply devise an algorithm for f$_{KD}$-$\mathcal{SHOIQ}^+$. However, this can be done easily by minor extensions of the already presented f$_{KD}$-$\mathcal{SHOIQ}$ algorithm.

Before presenting the algorithm we show that $\tau(\Sigma)$ is equisatisfiable to Σ. First, however, we show an auxiliary lemma which extends the results for transitive roles shown in [50]; see appendix for the proof.

Lemma 5 *Let $\mathcal{R}_h$ be a role hierarchy, let S be a role in $\mathcal{R}_h$, let $\mathcal{B}_S$ be the automaton for S w.r.t. $\mathcal{R}_h$ and let $w \in L(\mathcal{B}_S)$, where $w = R_1 \dots R_m$. If $\mathcal{I}$ satisfies $\mathcal{R}_h$, then $\mathcal{I}$ satisfies $\forall S.C \sqsubseteq \forall R_1.(\forall R_2.(\dots(\forall R_m.C)))$.*

Theorem 1 *Let Σ be an f$_{KD}$-$\mathcal{SROIQ}$ KB. Σ is satisfiable if and only if $\tau(\Sigma)$ is satisfiable.*

Example 6 Consider the f$_{KD}$-$\mathcal{SROIQ}$ KB $\Sigma = \langle \emptyset, \langle \mathcal{R}_h, \emptyset\rangle, \mathcal{A}\rangle$, where is as defined in Example 5, while $\mathcal{A}$ is as follows:

$$\begin{aligned}\mathcal{A} = \{&((a,b) : R_1) \geq 0.4, ((b,c) : R_2) \geq 0.7, ((c,d) : S) \geq 0.8,\\ &(a : \forall S.C) \geq 0.7, (d : \neg C) \geq 0.6\}\end{aligned}$$

Σ is unsatisfiable: For any interpretation $\mathcal{I}$ that satisfies $\mathcal{A}$ we must have $R_1^{\mathcal{I}}(a^{\mathcal{I}},b^{\mathcal{I}}) \geq 0.4, R_2^{\mathcal{I}}(b^{\mathcal{I}},c^{\mathcal{I}}) \geq 0.7, S^{\mathcal{I}}(c^{\mathcal{I}},d^{\mathcal{I}}) \geq 0.8, (\forall S.C)^{\mathcal{I}}(a^{\mathcal{I}}) \geq 0.7$, and $(\neg C)^{\mathcal{I}}(d^{\mathcal{I}}) \geq 0.6$. From the latter we also get $C^{\mathcal{I}}(d) \leq 0.4$. Moreover, $\mathcal{I}$ must satisfy $\mathcal{R}_h$; hence we have the following:

- Due to $R_1R_2 \sqsubseteq R$ we must have $R^{\mathcal{I}}(a^{\mathcal{I}},c^{\mathcal{I}}) \geq \min(R_1^{\mathcal{I}}(a^{\mathcal{I}},b^{\mathcal{I}}), R_2^{\mathcal{I}}(b^{\mathcal{I}},c^{\mathcal{I}})) \geq 0.4$.
- Due to $R \sqsubseteq S$ we must have $S^{\mathcal{I}}(a^{\mathcal{I}},c^{\mathcal{I}}) \geq R^{\mathcal{I}}(a^{\mathcal{I}},c^{\mathcal{I}}) \geq 0.4$.
- Due to $SS \sqsubseteq S$ we must have $S^{\mathcal{I}}(a^{\mathcal{I}},d^{\mathcal{I}}) \geq \min(S^{\mathcal{I}}(a^{\mathcal{I}},c^{\mathcal{I}}), S^{\mathcal{I}}(c^{\mathcal{I}},d^{\mathcal{I}})) \geq 0.4$.

From the latter we also get $1 - S^{\mathcal{I}}(a^{\mathcal{I}},d^{\mathcal{I}})) \leq 0.6$; hence $\max(1 - S^{\mathcal{I}}(a^{\mathcal{I}},d^{\mathcal{I}}), C(d^{\mathcal{I}}) \leq 0.6$ which leads to a contradiction with $(\forall S.C)^{\mathcal{I}}(a^{\mathcal{I}}) \geq 0.7$.

Now we show how we can transform the input knowledge base into an equisatisfiable f$_{KD}$-$\mathcal{SHOIQ}$ KB for which the tableaux algorithm for f$_{KD}$-$\mathcal{SHOIQ}$ can be used to decide unsatisfiability.

The automaton $\mathcal{B}_S$ from Example 5 is the automaton for S w.r.t. $\mathcal{R}_h$. Using $\mathcal{B}_S$ and the transformation procedure from Definition 9 we first re-rewrite $(a : \forall R.C) \geq 0.7$ as $(a : i_S) \geq 0.7$ and then $\mathcal{T}$ is expanded to to contain the following axioms:

$$\begin{array}{llll} i_S \sqsubseteq i_R & i_R \sqsubseteq \forall R_1.s_1 & s_1 \sqsubseteq \forall R_2.f_R & f_R \sqsubseteq f_S \\ f_S \sqsubseteq i_S & i_S \sqsubseteq \forall S.f_S & f_S \sqsubseteq C & \end{array}$$

Then, the tableaux algorithm is applied: From $(a : i_S) \geq 0.7$ and axiom $i_S \sqsubseteq i_R$ the algorithm would infer $(a : i_R) \geq 0.7$. By the latter together with $i_R \sqsubseteq \forall R_1.s_1$ and $((a,b) : R_1) \geq 0.4$ we then get $(b : s_1) \geq 0.7$. By a similar reasoning at some point we will get $(c : f_R) \geq 0.7$ and then due to $f_R \sqsubseteq f_S$ also $(c : f_S) \geq 0.7$. Subsequently, by $f_S \sqsubseteq i_S$ we also get $(c : i_S) \geq 0.7$, while by the latter, by the axiom $i_S \sqsubseteq \forall S.f_S$ and by the assertion $((c,d) : S) \geq 0.8$ we get $(d : f_S) \geq 0.7$ and finally $(d : C) \geq 0.7$. It can be seen that the latter together with $(d : \neg C) \geq 0.6$ creates a clash. ⊣

Given the above, next, we only present a tableaux algorithm for f$_{KD}$-$\mathcal{SHOIQ}^+$ knowledge bases.

Definition 10 A completion-graph **G** for an f$_{KD}$-$\mathcal{SHOIQ}^+$ knowledge base Σ is similar to an f$_{KD}$-$\mathcal{SHOIQ}$ completion-graph but with the following modifications:

G is said to contain a *clash* if one of the clash conditions of f$_{KD}$-$\mathcal{SHOIQ}$ in Definition 3 are satisfied or if additionally one of the following holds for x, y nodes:

- $\mathsf{Dis}(R,S) \in \mathcal{R}_a$ and y is an R_{n_1}- and S_{n_2}-neighbour of x.
- $\mathsf{ASym}(R) \in \mathcal{R}_a$, y is an R_{n_1}-neighbour of x and x an R_{n_2}-neighbour of y.
- $\langle \neg U', n \rangle \in \mathcal{L}(\langle x,y \rangle)$, with $n < 1$ and U' the pseudo-universal role.

♢

Definition 11 A *tableaux algorithm* for f$_{KD}$-$\mathcal{SHOIQ}^+$ initialises a completion-graph **G** for an f$_{KD}$-$\mathcal{SHOIQ}^+$ KB in the same way as in f$_{KD}$-$\mathcal{SHOIQ}$, with the additional step:

4. For each pair of nominal nodes r_i, r_j in **G** add $\langle U', 1 \rangle$ in $\mathcal{L}(\langle r_i, r_j \rangle)$.

Finally, **G** is expanded using tableaux rules from Tables 3 and 7. ♢

Table 7: Tableaux rules for role properties

Rule		Description
Self	If	$\langle \exists R.\mathsf{Self}, n\rangle \in \mathcal{L}(x)$, x is not blocked and x is not R_n-neighbour with itself
	then	$\mathcal{L}(\langle x,x\rangle) := \mathcal{L}(\langle x,x\rangle) \cup \{\langle R,n\rangle\}$
Ref	If	$\mathsf{Ref}(R) \in \mathcal{R}_a$, x is not blocked and x is not R_1-neighbour with itself
	then	$\mathcal{L}(\langle x,x\rangle) := \mathcal{L}(\langle x,x\rangle) \cup \{\langle R,1\rangle\}$
$\neg$-Self	If	$\langle \neg\exists R.\mathsf{Self}, n\rangle \in \mathcal{L}(x)$, x is not blocked, and x is not $\neg R_n$-neighbour with itself
	then	$\mathcal{L}(\langle x,x\rangle) := \mathcal{L}(\langle x,x\rangle) \cup \{\langle \neg R,n\rangle\}$

7 Conclusions

Fuzzy Description Logics are well-established extensions of classical Description Logics for representing fuzzy (vague) knowledge in a formal machine understandable way. They have already been used in many research applications, like multimedia processing and retrieval [17,37,43], semantic portals [26], ontology matching [21], decision making [54] and negotiation [9].

Several reasoning algorithms for supporting inference services over fuzzy DL knowledge bases have been presented the last decade. Straccia [51] presented a tableaux-based reasoning algorithm for f$_{KD}$-$\mathcal{ALC}$, while later Stoilos et al. [50] extended this algorithm for the fuzzy DLs f$_{KD}$-$\mathcal{SI}$, f$_{KD}$-$\mathcal{SHIN}$ and then also for f$_{KD}$-$\mathcal{ALCIQ}$ [45]. In a different approach, Straccia and Bobillo et al. [53,6,11] presented reduction algorithms that translate fuzzy DLs based on the standard fuzzy operators to classical DLs. The motivation is to use already implemented and highly optimised classical DL reasoners to support reasoning over fuzzy DL knowledge bases. However, the translation introduces additional TBox axioms (usually complex general axioms) and preliminary evaluation has shown that performance is affected [16]. For fuzzy DLs that use different fuzzy operators than the standard fuzzy ones, Straccia and Bobillo presented several reasoning algorithms for expressive DLs like $\mathcal{ALCQI}$ [8,10], however, it was later shown that the algorithms are in general incomplete as the extended languages are undecidable [2,3,13,15]; decidability (and hence completeness) is ensured only under restrictions on the form of allowed TBoxes [4].

Consequently, direct tableaux-based algorithms for fuzzy DLs under the standard fuzzy operators are still relevant both from practical as well as from a theoretical point of view. Such logics provide a direct, effective, and flexible way of representing and handling degrees of truth while the reasoning procedures can be directly optimised [44]. Although tableaux algorithms for quite expressive fuzzy DLs have been presented there is still no reasoning algorithm for f$_{KD}$-$\mathcal{SHOIN}$ and f$_{KD}$-$\mathcal{SROIQ}$. In the current paper we attempt to fill this gap by studying and presenting a tableaux reasoning algorithm for these logics. First, we studied reasoning over nominals ($\mathcal{O}$). We provided the proper extensions of the $\{o\}$- and NN-rules [30] and proposed a new rule, namely $\{o\}_2$. To the best of our knowledge a correct algorithm for handling nominals in fuzzy DLs has not been presented before. This algorithm can support reasoning over fuzzy ontologies expressed under the fuzzy OWL language f$_{KD}$-OWL [49]. Subsequently, we extended the algorithm of f$_{KD}$-$\mathcal{SHOIQ}$ in order to provide reasoning support for fuzzy nominals. Fuzzy nominals have been proposed in [5] as a fuzzification of the nominal constructor, however, no direct tableaux algorithm was provided.

Finally, we extended the f$_{KD}$-$\mathcal{SHOIQ}$ algorithm even further in order to support reasoning in the fuzzy DL f$_{KD}$-$\mathcal{SROIQ}$. To achieve this firstly, we show how complex role inclusion axioms can be handled in such DLs using automata [32] and additional TBox axioms [18]. More precisely, on the one hand, we prove an extension of a central proposition in [32] which dictates that the automata constructed using the same techniques as in [32] provide a correct characterisation of the semantics of cRIAs also in f$_{KD}$-$\mathcal{SROIQ}$. On the other hand, we present a full translation of cRIAs into TBox axioms following the ideas in [18] for grammar logics, and we prove that this yields an equisatisfiable f$_{KD}$-$\mathcal{SHOIQ}^+$ knowledge, i.e., an f$_{KD}$-$\mathcal{SROIQ}$ KB with an empty role hierarchy. Hence, then we only give a tableaux algorithm for f$_{KD}$-$\mathcal{SHOIQ}^+$ which extends the one for f$_{KD}$-$\mathcal{SHOIQ}$ to handle the universal role and the additional role properties of f$_{KD}$-$\mathcal{SROIQ}$.

As far as future directions are concerned we believe that the most important issue related to reasoning with fuzzy DLs is to implement and optimise such algorithms as well as evaluate them over well-known real-world ontologies.

A Omitted Proofs

Lemma 1 *An f$_{KD}$-$\mathcal{SHOIQ}$ knowledge base Σ is satisfiable iff there exists a fuzzy tableau T for Σ.*

Proof The proof is similar to the one for f$_{KD}$-$\mathcal{SHIN}$ [50], with the addition of nominals, qualified number restrictions and GCIs. Thus, in the following we only illustrate the different cases.

For the 'if' direction, let $T = (\mathbf{S}, \mathcal{L}, \mathcal{E}, \mathcal{V})$ be a fuzzy tableau for Σ; then, we can construct a model $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ of Σ in a similar way as in [50] by setting $\Delta^{\mathcal{I}} = \mathbf{S}$, $a^{\mathcal{I}} = \mathcal{V}(a)$, for each $a \in \mathbf{I}_\Sigma$, $\top^{\mathcal{I}}(s) = \mathcal{L}(s,\top)$ and $\bot^{\mathcal{I}}(s) = \mathcal{L}(s,\bot)$ for all $s \in \mathbf{S}$, while for concepts and roles we have the following:

$$
\begin{aligned}
B^{\mathcal{I}}(s) &= \mathcal{L}(s,B), \text{ for all } s \in \mathbf{S} \text{ and } B \in \mathbf{C} \cup \{\{o\} \mid o \in \mathbf{I}_\Sigma\} \\
R^{\mathcal{I}}(s,t) &= \begin{cases} R^+_{\mathcal{E}}(s,t) & \text{if } \mathsf{Trans}(R) \\ \max\limits_{P \,\underline{\circledast}\, R, P \neq R} (\mathcal{E}(R,\langle s,t\rangle), P^{\mathcal{I}}(s,t)) & \text{otherwise} \end{cases}
\end{aligned}
$$

where $R_{\mathcal{E}}(s,t) = \mathcal{E}(R,\langle s,t\rangle)$, for all $\langle s,t\rangle \in \mathbf{S}\times\mathbf{S}$, and $R^+_{\mathcal{E}}$ represents its sup-min transitive closure of $R_{\mathcal{E}}$ [33].

As with f$_{KD}$-$\mathcal{SHIN}$ [50], Properties 1 and 2 ensure the correct interpretations of the top and bottom concepts, Properties 9 and 10 the correct interpretation of inverse roles and role hierarchies, respectively, Property 16 (due to the results in [48,35]) that $\mathcal{I}$ is a model of the TBox, while Properties 17 and 18 that $\mathcal{I}$ is a model of $\mathcal{A}$. Additionally to f$_{KD}$-$\mathcal{SHIN}$ [50], Property 14 ensures that nominals are interpreted as singleton sets, Property 15 that the membership degree of elements in nominals is in accordance to their semantics, and Property 19 that nominals corresponding to ABox individuals are also interpreted correctly. Then, by induction on the structure of concepts we can show that for all $s \in \mathbf{S}$, $\mathcal{L}(s,C) \geq n$ implies $C^{\mathcal{I}}(s) \geq n$—similarly for $\mathcal{L}(s,C) > n$. We only show the case of nominals and qualified cardinality restrictions, which have not been presented before [50]:

- If $\mathcal{L}(s,\{o\}) \geq n$, then by Property 15 and construction of $\mathcal{I}$, $\{o\}^{\mathcal{I}}(s) = \mathcal{L}(s,\{o\}) \geq 1$.
- If $\mathcal{L}(s, \geq pR.C) \geq n$ then by Property 11 and definition of R^T, there are p elements t_i, s.t. $\mathcal{E}(R,\langle s,t_i\rangle) \geq n$, and $\mathcal{L}(t_i,C)$, $1 \leq i \leq p$. By construction, $R^{\mathcal{I}}(s,t_i) \geq n$ and by induction hypothesis $C^{\mathcal{I}}(t_i) \geq n$, thus
$$n \leq \sup_{t_i \in \Delta^{\mathcal{I}}} \{\ldots, \min_{i=1}^{p} \{\min(R^{\mathcal{I}}(s,t_i), C(t_i))\}, \ldots\} = (\geq pR.C)^{\mathcal{I}}(s).$$
- If $\mathcal{L}(s, \leq pR.C) \geq n$, then by Property 12 $\sharp R^T(s,>,1-n,C) \leq p$, i.e. there are at most p elements t_i such that, $\mathcal{E}(R,\langle s,t_i\rangle) > 1-n$, and $\mathcal{L}(t_i,C) > 1-n$ $1 \leq i \leq p$. Nevertheless, we need to show that this is the case also in $\mathcal{I}$, i.e. that for the set $R^{\mathcal{I}}(s,>,1-n,C) = \{x \in \Delta^{\mathcal{I}} \mid R^{\mathcal{I}}(s,x) > 1-n \text{ and } C^{\mathcal{I}}(x) > 1-n\}$ it holds $\sharp R^{\mathcal{I}}(s,>,1-n,C) \leq p$. Assume otherwise, i.e. that there is t_{p+1} different from all other t_i and such that $R^{\mathcal{I}}(s,t_{p+1}) > 1-n$ and $C^{\mathcal{I}}(t_{p+1}) > 1-n$. By construction, and since R must be a simple role, then $R^{\mathcal{I}}(s,t_{p+1}) = \mathcal{E}(R,\langle s,t_{p+1}\rangle) > 1-n$. Consequently, in order to have $\sharp R^T(s,>,1-n,C) \leq p$ it must be the case that $\mathcal{L}(t_{p+1},C) \leq 1-n$. But then by Property 13, $\mathcal{L}(t_{p+1},\neg C) \geq n$ must hold and by induction hypothesis $(\neg C)^{\mathcal{I}}(t_{p+1}) \geq n$, i.e. $C^{\mathcal{I}}(t_{p+1}) \leq 1-n$, which contradicts the original assumption that $C^{\mathcal{I}}(t_{p+1}) > 1-n$.

For the converse, let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a (*witnessed*) model for Σ; then, a fuzzy tableau $T = (\mathbf{S}, \mathcal{L}, \mathcal{E}, \mathcal{V})$ for Σ can be defined by setting $\mathbf{S} = \Delta^{\mathcal{I}}$, $\mathcal{E}(R,\langle s,t\rangle) = R^{\mathcal{I}}(s,t)$, $\mathcal{L}(s,C) = C^{\mathcal{I}}(s)$ and $\mathcal{V}(a) = a^{\mathcal{I}}$. It is easy to show that all properties in Definition 2 are satisfied as a direct consequence of the semantics of f$_{KD}$-$\mathcal{SHOIQ}$-concepts and since $\mathcal{I}$ is a witnessed model of Σ. □

Lemma 2 *Let Σ be an f_{KD}-$\mathcal{SHOIQ}$ knowledge base. Then,*

1. ***Termination**: when started for Σ the tableaux algorithm terminates.*
2. ***Correctness**: Σ has a fuzzy tableau if and only if the tableaux algorithm for f_{KD}-$\mathcal{SHOIQ}$ can be applied to Σ such that it yields a complete and clash-free completion-graph.*

Proof The termination of the algorithm (claim 1) is a consequence of the same properties that ensure termination in the case of the crisp $\mathcal{SHOIQ}$ language [30]. In brief we have the following observations:

- All rules apart from the *shrinking rules*[7] strictly extend the completion-graph by adding new nodes and edges or extending their labels while neither remove nodes, edges or pairs from them.
- New nodes are added only by the *generating rules*[8] and each of these rule is applied at most once for a given concept in the label of a given node x. Even if a shrinking rule is applied, and merges an R-neighbour y of x into another node z, then $\mathcal{L}(y)$ is added into $\mathcal{L}(z)$, z 'inherits' all the inequalities from y, and either z is an R-neighbour of x (if x is a nominal node or y a successor of x) or x is removed from the graph by an application of $\mathsf{Prune}(x)$ (if x is a blockable node and x is a successor of y).
- Since nodes are labelled with nonempty subsets of $cl(\Sigma)$ and edges with subsets of $\mathbf{R}_\Sigma$, obviously there is a finite number of possible labellings for a pair of nodes and an edge, while also the membership degrees that appear in nodes is also finite as in the case of f_{KD}-$\mathcal{SI}$. More precisely, for a pair of nodes and an edge there are at most 2^{8mlk} possible labellings, where $k = |\mathbf{R}_\Sigma|$, $m = |cl(\Sigma)|$ and $l = |N^{\mathcal{A}}|$. Since a path on which nodes are blocked cannot become longer, paths are of length at most 2^{8mlk}.
- As it is shown in [30] for $\mathcal{SHOIQ}$, the number of nominal nodes is bounded. This is a consequence of the following facts. The *NN* can only be applied after a nominal has been added to the label of a blockable node x in a branch of one of the blockable trees rooted in a root node. Otherwise it is not possible that a blockable node has a nominal node as a successor which is required by the first condition of the rule. Now since x contains one of the initial nominals that exist in $\mathcal{T}$ or $\mathcal{A}$, and the $\{o\}_1$-rule is applied with highest priority, x is merged with an existing nominal node which contains some of the initials nominals. As a consequence of this merging, it is possible that the predecessor of x is merged into a nominal node n_1 (created by an application of the *NN* to x) by the shrinking rules (due to the pruning, this cannot happen to a successor of x). The merge of the predecessor of x occurs because the *NN*-rule adds $\leq mR$ to x together with m already conjugated successors. Hence, x has $m+1$ successors (m created from the rule and one predecessor) and the $\leq$-rule will be executed. Now n_1 either contains one nominal from the initial ones or only nominals created by the *NN*. Repeating this argument, it is possible that all ancestors of x are merged into nominal nodes. But, since the length of a path of blockable nodes is bounded this repeated merging is bounded. Finally, when the *NN* has been applied to a concept $(\leq pR)$, it can never be applied to $(\leq pR)$ again.

The proof of the second claim extends the proof of f_{KD}-$\mathcal{SHIN}$ [50], by using the techniques of $\mathcal{SHOIQ}$ [30].

For the 'if' direction we construct a fuzzy tableau $T = (\mathbf{S}, \mathcal{L}, \mathcal{E}, \mathcal{V})$ from a complete and clash free completion-graph $\mathbf{G}$ by unravelling blockable 'tree' parts of the graph due to the lack the finite model property. Formally, the construction is the following.

An individual in $\mathbf{S}$ corresponds to a *path* in $\mathbf{G}$. Moving down to blocked nodes and up to blocking ones we can define infinite such paths. More precisely, a *path* is a sequence of pairs of nodes of $\mathbf{G}$ of the form $p = [\frac{x_{a_0}}{x'_{a_0}}, \ldots, \frac{x_{a_n}}{x'_{a_n}}]$. For such a path we define $\mathsf{Tail}(p) := x_{a_n}$ and $\mathsf{Tail}'(p) := x'_{a_n}$. With $[p \mid \frac{x_{a_{n+1}}}{x'_{a_{n+1}}}]$, we denote the path $[\frac{x_{a_0}}{x'_{a_0}}, \ldots, \frac{x_{a_n}}{x'_{a_n}}, \frac{x_{a_{n+1}}}{x'_{a_{n+1}}}]$. The set $\mathsf{Paths}(\mathbf{G})$ is defined inductively as follows:

- For each blockable node x of $\mathbf{G}$ that is a successor of a nominal node or a root node, $[\frac{x}{x}] \in \mathsf{Paths}(\mathbf{G})$, and
- For a path $p \in \mathsf{Paths}(\mathbf{G})$ and a node z in $\mathbf{G}$:
  - if z is a successor of $\mathsf{Tail}(p)$ and z is not blocked, then $[p \mid \frac{z}{z}] \in \mathsf{Paths}(\mathbf{G})$, or
  - if y is a successor of $\mathsf{Tail}(p)$ and z blocks y, then $[p \mid \frac{z}{y}] \in \mathsf{Paths}(\mathbf{G})$

By construction, it follows that all nodes occurring in a path are blockable nodes. Moreover, if $p \in \mathsf{Paths}(\mathbf{G})$, then $\mathsf{Tail}(p)$ is not blocked; $\mathsf{Tail}(p) = \mathsf{Tail}'(p)$ iff $\mathsf{Tail}'(p)$ is not blocked and at last $\mathcal{L}(\mathsf{Tail}(p)) = \mathcal{L}(\mathsf{Tail}'(p))$.

---

[7] these are the rules $\leq$, $\{o\}_1$ and $\leq_{\{o\}}$.

[8] these are the rules $\exists$, $\geq$ and *NN*.

Table 8: Fuzzy tableaux construction from complete and clash-free completion-graph

$$
\begin{aligned}
\mathbf{S} &= \mathsf{Nom}(\mathbf{G}) \cup \mathsf{Paths}(\mathbf{G}),\\
\mathcal{L}(p,C) &= \begin{cases} glb\{n \mid \langle C,n\rangle \in \mathcal{L}(\mathsf{Tail}(p))\}, & \text{if } p \in \mathsf{Paths}(\mathbf{G})\\ glb\{n \mid \langle C,n\rangle \in \mathcal{L}(p)\}, & \text{if } p \in \mathsf{Nom}(\mathbf{G})\end{cases}\\
\mathcal{L}(p,\neg A) &= 1-\mathcal{L}(p,A),\ \langle \neg A,n\rangle \in \mathcal{L}(\mathsf{Tail}(p)),\\
\mathcal{L}(p,\bot) &= 0, \text{ for all } p \in \mathbf{S},\\
\mathcal{L}(p,\top) &= 1, \text{ for all } p \in \mathbf{S},\\
\mathcal{E}(R,\langle p,[p \mid \tfrac{x}{x'}]\rangle) &= glb\{n \mid \langle R,n\rangle \in \mathcal{L}(\langle \mathsf{Tail}(p),x'\rangle)\}\\
\mathcal{E}(R,\langle [q \mid \tfrac{x}{x'}],q\rangle) &= glb\{n \mid \langle \mathsf{Inv}(R),n\rangle \in \mathcal{L}(\langle \mathsf{Tail}(q),x'\rangle)\}\\
\mathcal{E}(R,\langle p,x\rangle) &= glb\{n \mid x \in \mathsf{Nom}(\mathbf{G}) \text{ is an } R_n\text{-neighbour of } \mathsf{Tail}(p)\}\\
\mathcal{E}(R,\langle x,p\rangle) &= glb\{n \mid \mathsf{Tail}(p) \text{ is an } R_n\text{-neighbour of } x \in \mathsf{Nom}(\mathbf{G})\}\\
\mathcal{E}(R,\langle [\tfrac{x}{x}],[\tfrac{y}{y}]\rangle) &= glb\{n \mid x,y \text{ are root or nominal nodes and } y \text{ an } R_n\text{-neighbour of } x\},\\
\mathcal{E}(\neg R,\langle p,q\rangle) &= 1-\mathcal{E}(R,\langle p,q\rangle) \text{ for all } \langle p,q\rangle \in \mathbf{S}\times\mathbf{S},\\
\mathcal{V}(a_i) &= \begin{cases} [\tfrac{x_{a_i}}{x_{a_i}}] \text{ if } x_{a_i} \text{ is a root node in } \mathbf{G} \text{ with } \mathcal{L}(x_{a_i}) \neq \emptyset\\ [\tfrac{x_{a_j}}{x_{a_j}}] \text{ if } \mathcal{L}(x_{a_i}) = \emptyset, x_{a_j} \text{ a root node, with } \mathcal{L}(x_{a_j}) \neq \emptyset \text{ and } x_{a_i} \doteq x_{a_j}\end{cases}
\end{aligned}
$$

Secondly, we make use of the technique introduced in [51] in order to compute the degree that (pairs of) elements of the fuzzy tableau will belong to a concept (role). The function that calculates this degree is called *glb*. Roughly speaking for a specific path p this function is defined as the maximum of the set $\{n \mid \langle A,n\rangle \in \mathcal{L}(x)\} \cup \{0\}$. Due to normalization there may exist values of the form $n+\varepsilon$. It is important to note that an adequately small and carefully selected value ε must be chosen. For example, if $\langle \neg A, 0.19+\varepsilon\rangle \in \mathcal{L}(x)$ and $\langle A, 0.8+\varepsilon\rangle \in \mathcal{L}(x)$, then it should hold that $\varepsilon \leq 1-(0.8+0.19) = 0.01$. A similarly important case is if $\langle \forall R.C, 0.8\rangle \in \mathcal{L}(x)$ and $\langle R, 0.19+\varepsilon\rangle \in \mathcal{L}(\langle x,y\rangle)$. The existence of such a value is ensured by the clash-freeness of $\mathbf{G}$.

Next we use $\mathsf{Nom}(\mathbf{G})$ for the set of nominal and root nodes in $\mathbf{G}$, and define a tableau T as in Table 8.

It can be shown that T is a fuzzy tableau for Σ:

- Properties 1–3 of Definition 2 are satisfied because $\mathbf{G}$ is clash-free and due to the construction of T. Let $\mathcal{L}(p,\neg A) = n_1 \geq n$ and $\mathcal{L}(p,A) = n_2$. The definition of T and since we only consider non-zero assertions this implies that $\{\langle \neg A,n_1\rangle, \langle A,n_2\rangle\} \subseteq \mathcal{L}(\mathsf{Tail}(p))$. Since $\mathbf{G}$ is clash-free $n_1+n_2 \leq 1 \Rightarrow n_2 \leq 1-n_1$. Consequently, $\mathcal{L}(p,A) \leq 1-n_1 \leq 1-n$
- Properties 4 and 5 of Definition 2 are satisfied because none of the $\sqcup$, $\sqcap$ rules apply to any node in $\mathbf{G}$, and $\mathsf{Tail}(p)$ is not blocked. For example, let $\mathcal{L}(p, C\sqcap D) = n_1 \geq n$. The definition of T implies that, either $\langle C\sqcap D, n_1\rangle \in \mathcal{L}(\mathsf{Tail}(p))$ or $\langle C\sqcap D, n'+\varepsilon\rangle \in \mathcal{L}(\mathsf{Tail}(p))$, with $n_1 = n'+\varepsilon$. Completeness of $\mathbf{G}$ implies that either $\langle C,n_1\rangle \in \mathcal{L}(\mathsf{Tail}(p))$ and $\langle D,n_1\rangle \in \mathcal{L}(\mathsf{Tail}(p))$ or $\langle C,n'+\varepsilon\rangle \in \mathcal{L}(\mathsf{Tail}(p))$ and $\langle D,n'+\varepsilon\rangle \in \mathcal{L}(\mathsf{Tail}(p))$. Hence, $\mathcal{L}(s,C) \geq \mathcal{L}(s,C\sqcap D) \geq n$, $\mathcal{L}(s,D) \geq \mathcal{L}(s,C\sqcap D) \geq n$ Property 5 follows for similar reasons.
- For Property 6, let $p,q \in \mathbf{S}$ with $\mathcal{L}(p,\forall R.C) = n_1 \geq n$ and $\mathcal{E}(\neg R,\langle p,q\rangle) \ngeq n$. The definition of T implies that either $\langle \forall R.C, n_1\rangle \in \mathcal{L}(z_p)$ or $\langle \forall R.C, n'+\varepsilon\rangle \in \mathcal{L}(z_p)$ with $n_1 = n'+\varepsilon$ and $z_p = \mathsf{Tail}(p)$ if $p \in \mathsf{Paths}(\mathbf{G})$, or $z_p = z$ if $z \in \mathsf{Nom}(\mathbf{G})$. Now we have the following cases,
  - If $p \in \mathsf{Paths}(\mathbf{G})$, then $z_p = \mathsf{Tail}(p)$ and
    - If $q = [p \mid \frac{x}{x'}]$, then x' is an R_r-successor of $\mathsf{Tail}(p)$ and, since *glb* does not create unnecessary conjugations we have that $\langle R,r\rangle \in \mathcal{L}(\langle \mathsf{Tail}(p),x'\rangle)$ is such that it conjugates with $\langle \neg R,n\rangle$. Hence, due to completeness of $\mathbf{G}$ we have either $\langle C,n_1\rangle \in \mathcal{L}(x')$ or $\langle C,n'+\varepsilon\rangle \in \mathcal{L}(x')$, and either $x' = x$ or the blocking condition implies $\mathcal{L}(x') = \mathcal{L}(x) = \mathcal{L}(q)$.
    - If $p = [q \mid \frac{x}{x'}]$, then x' is an $\mathsf{Inv}(R)_r$-successor of $\mathsf{Tail}(q)$ and again, the definition of *glb* implies that $\langle \mathsf{Inv}(R),r\rangle \in \mathcal{L}(\langle \mathsf{Tail}(q),x'\rangle)$ conjugates with $\langle \neg\mathsf{Inv}(R),n\rangle$. Thus, due to completeness of $\mathbf{G}$, either $\langle C,n_1\rangle \in \mathcal{L}(\mathsf{Tail}(q)) = \mathcal{L}(q)$ or $\langle C,n'+\varepsilon\rangle \in \mathcal{L}(\mathsf{Tail}(q)) = \mathcal{L}(q)$.
    - If $p = [\frac{x}{x}]$ and $p = [\frac{y}{y}]$ for two root nodes x,y then y is an R_r-neighbour of x, and since the $\forall$-rule does not apply we have that either $\langle C,n_1\rangle \in \mathcal{L}(y) = \mathcal{L}(q)$ or $\langle C,n'+\varepsilon\rangle \in \mathcal{L}(y) = \mathcal{L}(q)$.

    - If $q = x \in \mathsf{Nom}(\mathbf{G})$, then x is an R_n-neighbour of $\mathsf{Tail}(p)$ and completeness implies that either $\langle C, n\rangle \in \mathcal{L}(x)$ or $\langle C, n' + \varepsilon\rangle \in \mathcal{L}(x)$.
  - If $z_p = z \in \mathsf{Nom}(\mathbf{G})$, then either
    - $q \in \mathsf{Paths}(\mathbf{G})$, then $\mathsf{Tail}(q)$ is an R_r-neighbour of z and completeness implies that either $\langle C, n\rangle \in \mathcal{L}(q)$, or $\langle C, n' + \varepsilon\rangle \in \mathcal{L}(x)$ or
    - $q = x \in \mathsf{Nom}(\mathbf{G})$, then x is an R_r-neighbour of z and completeness implies that either $\langle C, n\rangle \in \mathcal{L}(q)$ or $\langle C, n' + \varepsilon\rangle \in \mathcal{L}(x)$.

  The same proof applies for Property 6 with $\mathcal{L}(p, \forall R.C) > n$ and for Property 8.

- For Property 7 consider some $p \in \mathbf{S}$ with $\mathcal{L}(p, \exists R.C) \geq n$.
  - If $p \in \mathsf{Paths}(\mathbf{G})$, then either $\langle \forall R.C, n_1\rangle \in \mathcal{L}(p)$ or $\langle \forall R.C, n' + \varepsilon\rangle \in \mathcal{L}(p)$ with $n_1 = n' + \varepsilon$. Since $\mathsf{Tail}(p)$ is not blocked, completeness of $\mathbf{G}$ implies the existence of an R-neighbour y of $\mathsf{Tail}(p)$ with either $\langle C, n_1\rangle \in \mathcal{L}(y)$ or $\langle C, n' + \varepsilon\rangle \in \mathcal{L}(y)$.
    - If y is a nominal node, then $y \in \mathbf{S}$, $\mathcal{L}(y, C) \geq n$ and $\mathcal{E}(R, \langle p, y\rangle) \geq n$.
    - If y is a blockable node and a successor of $\mathsf{Tail}(p)$, then $\langle p, [p \mid \frac{y'}{y}]\rangle \in \mathbf{S}$, and either $y' = y$, or y' blocks y. In both cases either $\langle C, n_1\rangle$ or $\langle C, n' + \varepsilon\rangle$ are in $\mathcal{L}(y')$.
    - If y is a blockable node and a predecessor of $\mathsf{Tail}(p)$, then either $p = [r \mid \frac{y}{y} \mid \frac{\mathsf{Tail}(p)}{\mathsf{Tail}'(p)}]$, or $p = [r \mid \frac{z}{y} \mid \frac{\mathsf{Tail}(p)}{\mathsf{Tail}'(p)}]$ and $\mathsf{Tail}(p)$ blocks $\mathsf{Tail}'(p)$, hence $\mathsf{Tail}'(p)$ is an R-successor of z. In the first case either $\langle C, n_1\rangle \in \mathcal{L}(y)$ or $\langle C, n' + \varepsilon\rangle \in \mathcal{L}(y)$, while in the second case due to pair-wise blocking $\mathcal{L}(y) = \mathcal{L}(z)$.
  - If $p \in \mathsf{Nom}(\mathbf{G})$, then completeness implies the existence of some R-successor x of p with either $\langle C, n_1\rangle \in \mathcal{L}(x)$ or $\langle C, n' + \varepsilon\rangle \in \mathcal{L}(x)$
    - If x is a nominal node, then $\mathcal{E}(R, \langle p, x\rangle) \geq n$ and $\mathcal{L}(x, C) \geq n$.
    - If x is a blockable node, then x is a safe R-neighbour of p and thus not blocked. Hence, there is a path $q \in \mathsf{Paths}(\mathbf{G})$ with $\mathsf{Tail}(q) = x$, $\mathcal{E}(R, \langle p, q\rangle) \geq n$ and $\mathcal{L}(q, C) \geq n$.

  In any of these cases, $\mathcal{E}(R, \langle p, q\rangle) \geq n_1 \geq n$, $\mathcal{L}(q, C) \geq n_1 \geq n$. Similarly for $\mathcal{L}(p, \exists R.C) > n$.

- Property 9 in Definition 2 is satisfied due to the symmetric definition of $\mathcal{E}$.
- Property 10 in Definition 2 is satisfied due to the definition of the R-successor that takes into account the role hierarchy $\underline{\sqsubseteq}^{*}$.
- Properties 11–13 in Definition 2 are satisfied due to the construction of T as in the classical case [30].
- Properties 14 and 15 are due to completeness of $\mathbf{G}$, the fact that nominal nodes are not 'unravelled', while Property 19 due to initialisation of $\mathbf{G}$.
- Property 16 is due to completeness of $\mathbf{G}$.
- Properties 17–18 are satisfied cause of the initialisation of the completion-graph and the fact that the algorithm never blocks root nodes. Furthermore, for each root node x_{a_i} whose label and edges are removed by the Merge method, there is another root node x_0^j with $x_{a_i} = x_{a_j}$ and $\{\langle C, n\rangle \mid (a_i : C) \geq n \in \mathcal{A}\} \subseteq \mathcal{L}(x_{a_j})$.

For the 'only-if' direction, if $T = (\mathbf{S}, \mathcal{L}, \mathcal{E}, \mathcal{V})$ is a fuzzy tableau for Σ we can use T, to guide the application of the expansion rules such that they yield a completion-graph $\mathbf{G}$ that is both complete and clash-free. More precisely, we can define a mapping π from nodes in the completion-graph to individuals in $\mathbf{S}$ of the tableau and use this π to modify the non-deterministic rules in such a way that we always choose the 'correct' fuzzy pair to be added in the label of some node [50]. This, together with the termination property ensure that at some point blocking will occur and the resulting completion-graph would also be clash-free. □

Proposition 2 *Let $\mathcal{R}_h$ be a regular role hierarchy, let $\mathcal{B}_S$ be the automaton constructed for a role S w.r.t. $\mathcal{R}_h$ according to the method in [32], and let $L(\mathcal{B}_S)$ denote the language accepted by $\mathcal{B}_S$. Then, $\mathcal{I}$ is a model of $\mathcal{R}_h$ if and only if for each role S occuring in $\mathcal{R}_h$, each word $w \in L(\mathcal{B}_S)$ and each $w^{\mathcal{I}}(a, b) \geq n$ we have $S^{\mathcal{I}}(a, b) \geq n$.*

Proof The 'if' direction is similar to the one in [32] (i.e., by showing the contrapositive). More precisely, assume that $\mathcal{I}$ is not a model of $\mathcal{R}_h$. Then, there exists a cRIA $w \sqsubseteq S \in \mathcal{R}_h$ not satisfied by $\mathcal{I}$. Hence, there are $\langle x, y\rangle$ and $n \in (0, 1]$ such that $w^{\mathcal{I}}(x, y) = n$ and $S^{\mathcal{I}}(x, y) < n$. But, since $\mathcal{B}_S$ is constructed by the method in [32] then it satisfies Lemma 12.1 in [32], i.e., $w \in L(\mathcal{B}_S)$.

The 'only-if' direction is again proved by similar techniques as in [32] but considering that the semantics of cRIAs are given by sup-t composition. More precisely, let $\mathcal{I}$ be a model of $\mathcal{R}_h$, let S be a role, let $w \in L(\mathcal{B}_S)$

and let $w^{\mathcal{I}}(a,b)=n$. We need to show that $S^{\mathcal{I}}(a,b)\geq n$. Since $\mathcal{R}_h$ is regular there exists a strict partial order $\prec$ such that each cRIA in $\mathcal{R}_h$ is $\prec$-regular. Hence, we can use well-founded induction over $\prec$.

First, note that $w\in L(\mathcal{B}_S)$ induces a decomposition $w=w_1\dots w_k$ and word $\hat{w}=S_1\dots S_k$ such that

- $S_i\prec S$ or $S_i=S$ for all $1\leq i\leq k$
- $\hat{w}\in L(\hat{\mathcal{A}}_S)$,[9] and
- $w_i\in L(\mathcal{B}_{S_i})$

Moreover, $w^{\mathcal{I}}(a,b)=n$ implies that there exist k x_i with $a=x_0$, $b=x_k$, $w^{\mathcal{I}}_{i+1}(x_i,x_{i+1})=n_{i+1}$, and $n=\min(n_1,\dots,n_k)$ for $0\leq i<k$. By induction hypothesis, $S_i^{\mathcal{I}}(x_i,x_{i+1})\geq n_{i+1}$ and $\hat{w}^{\mathcal{I}}(a,b)=n$. By case analysis on the form of axioms in $\mathcal{R}_h$ that have S in the right-hand side it can be shown that $S^{\mathcal{I}}(a,b)\geq n$. We show one case, as the rest follow similarly (cf. also [32]).

If $SS\sqsubseteq S\notin\mathcal{R}_h$ and $S^-\sqsubseteq S\notin\mathcal{R}_h$, then, by construction of $\hat{\mathcal{A}}_S$, $\hat{w}$ is of the form

$$\hat{w}=u_1\dots u_m x v_1\dots v_\ell \quad\text{and}\quad \begin{array}{l} u_iS\sqsubseteq S\in\mathcal{R}_h,\text{ for each }1\leq i\leq m\\ x\sqsubseteq S\in\mathcal{R}_h\text{ or }x=S\\ Sv_j\sqsubseteq S\in\mathcal{R}_h,\text{ for each }1\leq j\leq\ell\end{array}$$

Since $\mathcal{I}$ is a model of $\mathcal{R}_h$ it satisfies the above axioms and hence it follows that $S^{\mathcal{I}}(a,b)\geq\hat{w}^{\mathcal{I}}(a,b)=n$. □

Lemma 5 *Let $\mathcal{R}_h$ be a role hierarchy, let S be a role in $\mathcal{R}_h$, let $\mathcal{B}_S$ be the automaton for S w.r.t. $\mathcal{R}_h$ and let $w\in L(\mathcal{B}_S)$, where $w=R_1\dots R_m$. If $\mathcal{I}$ satisfies $\mathcal{R}_h$, then $\mathcal{I}$ satisfies $\forall S.C\sqsubseteq\forall R_1.(\forall R_2.(\dots(\forall R_m.C)))$.*

Proof Let $a_i\in\Delta^{\mathcal{I}}$ for $0\leq i\leq m$, where $a_0=a$ be objects in $\Delta^{\mathcal{I}}$, and let $(\forall S.C)^{\mathcal{I}}(a)\geq n$. By Proposition 2 we have that $S^{\mathcal{I}}(a_0,a_m)\geq\min(R_1^{\mathcal{I}}(a_0,a_1),\dots R_m^{\mathcal{I}}(a_{n-1},a_n))$, hence we also have $c(S^{\mathcal{I}}(a_0,a_m))\leq c(\min(R_1^{\mathcal{I}}(a_0,a_1),\dots R_m^{\mathcal{I}}(a_{n-1},a_n)))$. Then, we have the following equivalences:

$$(1)\ \inf_{a_m\in\Delta^{\mathcal{I}}}\max(c(S^{\mathcal{I}}(a_0,a_m)),C^{\mathcal{I}}(a_m))\geq n \qquad \Rightarrow_{\text{monotonicity}}$$

$$(2)\ \inf_{a_m\in\Delta^{\mathcal{I}}}\max(c(\min_{i=1}^{m} R_i^{\mathcal{I}}(a_{i-1},a_i)),C^{\mathcal{I}}(a_m))\geq n \qquad \Rightarrow_{\text{De Morgan}}$$

$$(3)\ \inf_{a_m\in\Delta^{\mathcal{I}}}\max(\max_{i=1}^{m} c(R_i^{\mathcal{I}}(a_{i-1},a_i)),C^{\mathcal{I}}(a_m))\geq n \qquad \Rightarrow_{\text{associativity}}$$

$$(4)\ \inf_{a_m\in\Delta^{\mathcal{I}}}\max(\max_{i=1}^{m-1} c(R_i^{\mathcal{I}}(a_{i-1},a_i)),\max(c(R_m^{\mathcal{I}}(a_{m-1},a_m)),C^{\mathcal{I}}(a_m)))\geq n \qquad \Rightarrow_{\text{Property }(\blacklozenge)}$$

$$(5)\ \max(\max_{i=1}^{m-1} c(R_i^{\mathcal{I}}(a_{i-1},a_i)),\inf_{a_m\in\Delta^{\mathcal{I}}}\max(c(R_m^{\mathcal{I}}(a_{m-1},a_m)),C^{\mathcal{I}}(a_m)))\geq n \qquad \Rightarrow$$

$$(6)\ \max(\max_{i=1}^{m-1} c(R_i^{\mathcal{I}}(a_{i-1},a_i)),(\forall R_m.C)^{\mathcal{I}}(a_{m-1}))\geq n$$

Now, note that a_{m-1} is an arbitrary object of $\Delta^{\mathcal{I}}$ and that fuzzy DLs under the standard fuzzy operators satisfy the witnessed model property; hence, from (6) we have $\inf_{a_{m-1}}\max(\max_{i=1}^{m-1} c(R_i^{\mathcal{I}}(a_{i-1},a_i)),(\forall R_m.C)^{\mathcal{I}}(a_{m-1}))\geq n$. Working similarly as above we can infer $\max(\max_{i=1}^{m-2} c(R_i^{\mathcal{I}}(a_{i-1},a_i)),(\forall R_{m-1}(\forall R_m.C))^{\mathcal{I}}(a_{m-2}))\geq n$. Consequently, after m steps we can infer $\forall R_1.(\forall R_2.(\dots(\forall R_m.C)))^{\mathcal{I}}(a)\geq n$. □

Theorem 1 *Let Σ be an f_{KD}-$\mathcal{SROIQ}$ KB. Σ is satisfiable if and only if $\tau(\Sigma)$ is satisfiable.*

Proof For the 'only-if' direction assume that $\mathcal{I}$ is a model of Σ but not of $\tau(\Sigma)$. By construction, $\tau(\Sigma)$ is almost like Σ but with some concepts of the form $\forall S.C$ replaced with fresh concepts of the form i_S and some additional axioms of the form $s_i\sqsubseteq\forall R_i.s_{i+1}$. All axioms of Σ that have not been modified in $\tau(\Sigma)$ are clearly satisfied by $\mathcal{I}$. Moreover, for those axioms of Σ where $\forall S.C$ has been re-written using i_S, since i_S is a fresh concept the model $\mathcal{I}$ can be trivially extended to $\mathcal{I}'$ which satisfies them. More precisely, if $(\forall S.C)^{\mathcal{I}}(a)=n$, then we can set $i_S^{\mathcal{I}'}(a)=n$. Hence, that $\mathcal{I}$ does not satisfy $\tau(\Sigma)$ it must be due to the newly added axioms $s_i\sqsubseteq\forall R_i.s_{i+1}$. However, again, since each s_i is a fresh concept, $\mathcal{I}'$ can be further extended to satisfy these

[9] Note that here we refer to notation from [32]: $\hat{\mathcal{A}}_S$ is the automaton constructed for S w.r.t. $\mathcal{R}_h$ by only considering cRIAs that have S in the right-hand side—that is, if $R_1R_2\sqsubseteq S\in\mathcal{R}_h$ and R_1 also has an automaton, then $\hat{\mathcal{A}}_S$ does not include the states of $\mathcal{B}_{R_1}$ but only those states that are necessary to capture $R_1R_2\sqsubseteq S$.

axioms. Thus, that $\mathcal{I}'$ is not a model it must be due to a set of axioms of the form $i_S \sqsubseteq \forall R_1.s_1, \ldots, s_n \sqsubseteq \forall R_m.f_S, f_S \sqsubseteq C$, i.e., that also involves the axiom with C. These axioms can be unfolded into a single axiom of the form $i_S \sqsubseteq \forall R_1.(\forall R_2.(\ldots(\forall R_m.C)))$, and finally to $\forall S.C \sqsubseteq \forall R_1.(\forall R_2.(\ldots(\forall R_m.C)))$. By construction of $\tau(\Sigma)$ it follows that $R_1 \ldots R_m \in L(\mathcal{B}_S)$, hence, by Lemma 5 $\mathcal{I}$ must satisfy the axiom which leads to a contradiction.

For the 'if' direction, assume that $\mathcal{I}$ is a model of $\tau(\Sigma)$. A model $\mathcal{I}'$ for Σ can be constructed as follows:

- $\Delta^{\mathcal{I}'} = \Delta^{\mathcal{I}}$
- For each individual a, $a^{\mathcal{I}'} = a^{\mathcal{I}}$
- For each atomic concept $A \in cl(\Sigma), A^{\mathcal{I}'}(a) = A^{\mathcal{I}}(a)$
- If R is minimal w.r.t. $\prec$, then $R^{\mathcal{I}'}(a,b) = R^{\mathcal{I}}(a,b)$
- If R is not minimal w.r.t. $\prec$, then

$$R^{\mathcal{I}'}(a,b) = \max(R^{\mathcal{I}}(a,b), \min(R_1^{\mathcal{I}}(a,x_1), R_2^{\mathcal{I}}(x_1,x_2), \ldots, R_m^{\mathcal{I}}(x_{m-1},b)))$$

where $R_i \prec S$ and there are axioms of the form $s_i \sqsubseteq \forall R_i.s_{i+1}$ in $\tau(\Sigma)$, with $1 \leq i \leq m$.

Note that the interpretation of complex roles is inductive. However, since $\mathcal{R}_h$ is regular the induction on $\prec$ is well-founded. By Proposition 2 and the inductive interpretation of complex roles using $\prec$ and $\mathcal{B}_S$ it follows that $\mathcal{I}'$ is a model of the role hierarchy of Σ. Moreover, by induction on the structure of concepts it also follows that $D^{\mathcal{I}}(a) = D^{\mathcal{I}'}(a)$. More precisely, if $D = C_1 \sqcap C_2$ and $(C_1 \sqcap C_2)^{\mathcal{I}}(a) = n$ we have $C_1^{\mathcal{I}}(a) = n_1$, $C_2^{\mathcal{I}}(a) = n_2$, with $n = \min(n_1, n_2)$. Then, by the induction hypothesis we have $C_1^{\mathcal{I}'}(a) = n_1$, $C_2^{\mathcal{I}'}(a) = n_2$ and hence $(C_1 \sqcap C_2)^{\mathcal{I}'}(a) = \min(n_1, n_2) = n$. All other cases follow in a similar way. The only non-trivial interesting case is if $D = \forall S.C, (\forall S.C)^{\mathcal{I}}(a) = n, S^{\mathcal{I}'}(a,b) = p$, and S is complex. If $1-p = n$ (i.e., $p = 1-n$), then $\max(1-p, C^{\mathcal{I}'}(b)) = n$ regardless of $C^{\mathcal{I}'}(b)$ and hence $(\forall S.C)^{\mathcal{I}'}(a) = n$. Assume that $p > n$. Since $(\forall S.C)^{\mathcal{I}}(a) = n$ we have that $i_S^{\mathcal{I}}(a) = n$. Now there are two cases:

- $S^{\mathcal{I}}(a,b) = p$. Since S is complex $\tau(\Sigma)$ contains $i_S \sqsubseteq \forall S.f_S$ and $f_S \sqsubseteq C$. Since $p > n$ and $\mathcal{I}$ is a model of $\tau(\Sigma)$ we must have $C^{\mathcal{I}}(b) = n$.
- $S^{\mathcal{I}}(a,b) \neq p$. Then, there exist m R_i with $R_1^{\mathcal{I}}(a,x_1) = p_1, \ldots R_m^{\mathcal{I}}(x_m,b) = p_m$ s.t. $p = \min(p_1, \ldots p_m)$. By construction of $\mathcal{I}'$ there are axioms $s_i \sqsubseteq \forall R_i.s_{i+1}$ and $s_{m+1} \sqsubseteq f_S, f_S \sqsubseteq C$, in $\tau(\Sigma)$. Moreover, since $p > n$, then also $p_i > n$ for each $1 \leq i \leq m$. But then, since $\mathcal{I}$ is a model of $\tau(\Sigma)$, for all s_i we must have $s_i^{\mathcal{I}}(x_i) = n$ and hence, also $C^{\mathcal{I}}(b) = n$.

In both cases, by induction hypothesis, $C^{\mathcal{I}'}(a) = n$, hence we finally get $(\forall S.C)^{\mathcal{I}'}(a) = n$.

Concluding, we need to show that $\mathcal{I}'$ satisfies each axiom $C \sqsubseteq D$. Since $\mathcal{I}$ is a model of $\tau(\Sigma)$ we have $C^{\mathcal{I}}(a) \leq D^{\mathcal{I}}(a)$. However, as shown $C^{\mathcal{I}'}(a) = C^{\mathcal{I}}(a) \leq D^{\mathcal{I}}(a) = D^{\mathcal{I}'}(a)$, hence also $\mathcal{I}$ satisfies $C \sqsubseteq D$. □

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